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NON-COMMENSURATE MANIPULATOR JACOBIAN

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Abstract:

The Jacobian matrix of a robot manipulator is central to the analysis, kinematics, dynamics, and control of robot manipulators. In many instances, the Jacobian and its inverse or pseudo-inverse are needed and utilized in the control equations of robot manipulators. In robotics, translations and rotations, transforms whose variables of motion, distance for translations, angles for rotations, combine to generate motions in a given workspace. Object motion and speed also combine units of angle and distance or angular and translational velocities. The mathematical complexities of the control process often obscure the interaction of units and lead to results that may be misinterpreted, erroneous, or simply arbitrary. The research results presented in this article indicate that control equations based on the manipulator Jacobian, its generalized inverse, or its pseudo-inverse may be erroneously combining quantities of different physical units thereby reaching arbitrary results.

Key Words: Jacobian, non-commensurate, units

1. Introduction

Systems with input and/or output vectors composed of quantities of different physical nature and, therefore, using different physical units are called non-commensurate systems when described by physically consistent equations [1][2]. Robotics offers an example of non-commensurate system in its basic joint rate control equation for robot manipulators:

$$V = J\dot{q} \tag{1}$$

where the velocity vector *V*, known as the twist vector, is composed of the linear velocity vector v, with units of distance/time, and the angular velocity vector w, with units of angle/time or

$$V = [v, w]^T$$

and the joint velocity vector $\dot{q}$, has elements of angular velocity (for revolute joints), and elements of linear velocity for prismatic joints. Physical consistency is achieved in Eq. (1) by proper choice of units for the elements of the system matrix *J*. This example indicates that the matrix *J* is often composed of elements with different physical units.

It has been shown that certain mathematical derivations concerning non-commensurate systems lead to inconsistent and erroneous results. As an example, in robotics, the use of singular value decomposition or eigen-values in the derivation of manipulability measures in robotics is invalid [2][3]. In cases where the task space and the joint space have different dimensions, the Jacobian matrix is not square and manipulator control requires the use of a generalized or pseudo inverse.

Several authors [4][5][6][7] have discussed already the problems associated with the use of the pseudo-inverse in solving for the joint rates given a desired twist vector. The pseudo-inverse is based on Euclidian norms of both the joint rate vector and the twist vector. However, the twist vector space is not Euclidian since its elements do not share the same physical unit. In the case of a mixed revolute/prismatic joint manipulator, the joint-rate vector space is not Euclidian either. This articles discusses the physical consistency (PC) of the manipulator Jacobian and its pseudo-inverse.

2. Inverse Velocity Kinematics

When the task and joint spaces are not equal, the Jacobian matrix is rectangular and Eq. 1 is solved by use of a generalized inverse.

Let $J=FC$ be a full rank factorization of J where F is a matrix of dimension (6,r) with full column rank r, and C has dimensions (r, n) with full row rank r, and n is the number of joints in the manipulator. The pseudo-inverse of J is given by [4]:

$$J^{\dagger} = C^T(CC^T)^{-1}(F^TF)^{-1}F^T = C^{\dagger}F^{\dagger} \qquad (2)$$

and the generalized inverse is:

$$\begin{aligned} J^{\#} &= [M_q^{-1}C^T(CM_q^{-1}C^T] \\ &\quad [(F^TM_vF)^{-1}F^TM_v] = C^{\#}F^{\#} \end{aligned} \qquad (3)$$

where M_q and M_v are positive definite metrics.

In the case where the desired twist V is in the range of the Jacobian matrix J, the residual $V - J\dot{q}$ is zero and the metric M_v is not needed, simplifying the expression for $J^{\#}$ to:

$$J^{\#} = [M_q^{-1}C^T(CM_q^{-1}C^T][(F^TF)^{-1}F^T]. \qquad (4)$$

If the Jacobian has full column rank, then the joint rates metric M_q is not needed and Eq. (3) becomes:

$$J^{\#} = (J^TM_vJ)^{-1}J^TM_v. \qquad (5)$$

When the Jacobian has full rank and the twist vector is in the range of J, no metrics are needed and the generalized inverse is equal to the pseudo inverse. However, all manipulators have singular configurations where V lies outside the range of J [8]. Therefore, every manipulator has configurations where metrics are needed.

Redundant manipulators in configurations where the Jacobian has full row rank have a generalized Jacobian inverse that does not require a twist metric. Also, if all joints are of the same kind, revolute or prismatic but not mixed, the joint space metric is not needed for physical consistency and the pseudo-inverse can be used. However, the joint space metric is still needed for invariance to rigid body transformations and scaling.

3. Physical consistency of the pseudo-inverse Jacobian

The physical consistency of the manipulator Jacobian pseudo-inverse is affected by rigid body transformations. The effect of rotations and translations on the physical consistency of the Jacobian pseudo-inverse are examined separately here.

<u>Rotations</u>:

Let ${}^iV$ and ${}^jV$ be expressions of a twist vector V in frame F_i and frame F_j respectively where F_j is a rotation of frame F_i (no translation) so that ${}^jV = {}^jG_i\,{}^iV$ where ${}^jG_i$ is the twist frame transform matrix from F_i to F_j. If the pseudo-inverse of the Jacobian is physically consistent in frame F_i, is it also consistent in a rotated frame F_j?
The Jacobian pseudo-inverse is given by Eq. (2) and it is physically consistent in F_i.

$${}^iJ^{\dagger} = C^T(CC^T)^{-1}(F^TF)^{-1}F^T,$$

Its expression in frame F_j is

$${}^jJ^{\dagger} = ({}^jG_i\,{}^iJ)^{\dagger} = [({}^jG_iF)C]^{\dagger} \qquad (6)$$

or

$${}^jJ^{\dagger} = C^T(CC^T)^{-1}\ F^T\,{}^jG_i^T\ {}^jG_iF)^{-1}F^T\ {}^jG_i^T \qquad (7)$$

which simplifies to:

$${}^jJ^{\dagger} = C^T(CC^T)^{-1}\ (F^TF)^{-1}F^T\ {}^iG_j \qquad (8)$$

since ${}^jG_i^T = ({}^jG_i)^{-1} = {}^iG_j$ for the case where ${}^jG_i$ is a rotation. Equation (8) is therefore

$${}^jJ^{\dagger} = {}^iJ^{\dagger}\ {}^iG_j. \qquad (9)$$

Partitioning the pseudo inverse into two n x 3 matrices, W and X for ${}^iJ^{\dagger}$, and Y and Z for ${}^jJ^{\dagger}$ i.e.:

$${}^iJ^{\dagger} = [W\ X] \qquad (10)$$

and

$${}^jJ^{\dagger} = [Y\ Z] = [WR\ \ XR] \qquad 11)$$

where $R = {}^iR_j$ is the rotation matrix of frame F_j from frame F_i.

Since ${}^iJ^{\dagger}$ operates on ${}^iV = [v, w]^T$, each component in a row of W (or a row of X) must have like units or have zero value. R is dimensionless therefore row elements of Y (or Z) must have the same units as row elements of W (or X) which makes ${}^{j}J^{\dagger}$ physically consistent. This proves that if a Jacobian pseudo inverse is physically consistent in one reference frame, then it is physically consistent in any rotated frame.

Translations

If a Jacobian pseudo-inverse matrix is physically consistent with respect to one link frame F_i, then it is not necessarily consistent with respect to a translated reference frame F_t. A simple example will demonstrate this fact. The peg-in-a-hole problem [4][9] with a simple 2 joint PR virtual manipulator [10] is described here. The PR manipulator has the DH-parameter description of Table 1.

Table 1. D-H parameter of PR manipulator

Joint	**d**	**a**	q	a	**type**
1	d_1	0	0	0	Prismatic
2	d_2	0	θ_2	0	Revolute

Figure 1 illustrates the peg-in-the hole situation for this manipulator. The Jacobian in frame F_2 is given by

$$
{}^{2}J = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{T} \tag{12}
$$

and the pseudo inverse ${}^{2}J^{\dagger} = {}^{2}J^{T}$ is physically consistent. In an arbitrarily translated frame F_t (with no rotation) the Jacobian is

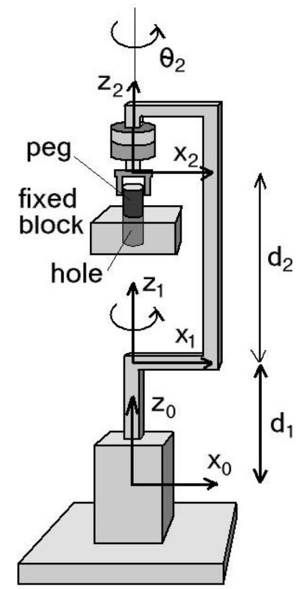
$$
{}^{t}J = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ p_y & -p_x & 0 & 0 & 0 & 1 \end{bmatrix}^{T} \tag{13}
$$

obtained by left-multiplying ${}^{2}J$ by the generalized frame transform ${}^{2}G^{t,2}$ given by

$$
{}^{2}G^{t,2} = \begin{bmatrix} I_3 & [p\times] \\ [0]_{3,3} & I_3 \end{bmatrix}
= \begin{bmatrix} 1 & 0 & 0 & 0 & -p_z & p_y \\ 0 & 1 & 0 & p_z & 0 & -p_x \\ 0 & 0 & 1 & -p_y & p_x & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}
$$

where $\mathbf{p} = \begin{bmatrix} p_x & p_y & p_z \end{bmatrix}^{T}$ is the translation vector.

Figure 1. Peg-in-hole with PR arm



The pseudo-inverse of the Jacobian given in Eq. (13) is computed as

$$
{}^{t}J^{\dagger} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ \dfrac{p_y}{1+p_x^2+p_y^2} & \dfrac{p_x}{1+p_x^2+p_y^2} & 0 & 0 & 0 & \dfrac{1}{1+p_x^2+p_y^2} \end{bmatrix}. \tag{15}
$$

The denominator in the elements of ${}^{t}J^{\dagger}$ is not physically consistent as it adds a unitless number to numbers with units of distance-squared. This shows that the Jacobian pseudo-inverse is not physically consistent when expressed in a translated reference frame. Consequently, any mathematical development that relies on the pseudo-inverse of the Jacobian in a translated frame will be physically inconsistent and will lead to ambiguous and misleading results.

4. Conclusion

Robotics control and research often relies on the use of the Jacobian matrix, its inverse, generalized inverse, or pseudo-inverse in the analysis and control of robot manipulators. The mathematical complexities of the processes employed obscure the interaction of physical units and sometimes lead to erroneous, misleading, or ambiguous results. This article discusses the physical consistency of the Jacobian matrix of robot manipulators and more precisely the manipulator Jacobian pseudo-inverse and shows that physical consistency is not always preserved when changing the frame of expression of the pseudo-inverse. While rotated frames do not affect the physical consistency of the pseudo-inverse, translated frames do not always preserve physical consistency.

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