Introduction to Complex Mathematics

1. Introduction

In our analysis of discrete-time signals and systems, complex numbers will play an incredibly important role; virtually everything we do from here on out will involve complex numbers in one form or another. Therefore, it is worthwhile, at this point, to do a short review of complex numbers. You can find additional discussions on complex numbers in Appendix A (pp, 378-98) of *DSP First: A Multimedia Approach* (J. H. McClellan, *et. al.*), the course textbook.

2. Introduction to complex numbers

A. Basics

Definition: The *imaginary number* $\mathbf{j}$ is defined as:

$$\mathbf{j} \equiv \sqrt{-1} \tag{1}$$

A consequence of definition (1) is that:

$$\mathbf{j}^2 = -1 \,. \tag{2}$$

Definition: A *complex number* z is defined as an ordered pair of real numbers x and y:

$$z \equiv (x, y) = x + \mathbf{j}y \tag{3}$$

such that the *real part* of z is given by:

$$Re[z] \equiv x \tag{4}$$

and the *imaginary part* of z is given by:

$$Im[z] \equiv y \,. \tag{5}$$

We can think of a complex number as a point in the *complex plane*, which consists of the horizontal real axis, and the imaginary vertical axis, as shown in Figure 1 below.

Definition: The *conjugate* z^* of a complex number z is defined as:

$$z^* = x - \mathbf{j}y \tag{6}$$

B. Polar form and Euler's formula

The complex number in equation (3) is in *rectangular* or *Cartesian* form. For our studies, oftentimes a more convenient form will be the *polar* form as shown in Figure 2, such that:

$$z = x + \mathbf{j}y = r\cos\theta + \mathbf{j}r\sin\theta \tag{7}$$

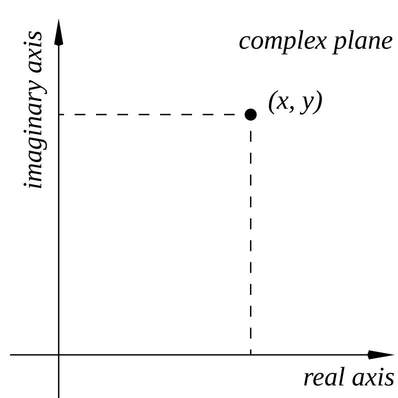

Figure 1

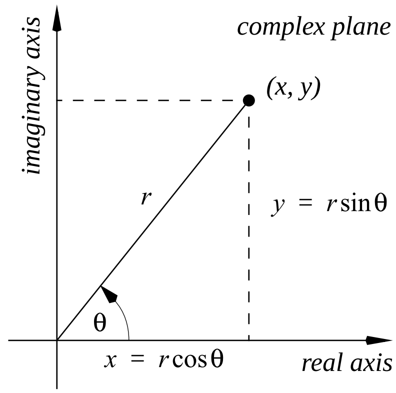

Figure 2

where

$$r = \sqrt{x^2 + y^2} \tag{8}$$

$$\theta = \operatorname{atan}(y, x) \tag{9}$$

Note that in equation (9) we use the two-argument inverse tangent function. Unlike the single-argument inverse tangent function $\operatorname{atan}(y/x)$, which returns angles in the range $[-\pi/2, \pi/2]$ (first and fourth quadrants only), the two-sided inverse tangent function returns angles in the range $[-\pi, \pi]$ (all quadrants). The following relationship exists between the two-argument and one-argument atan functions:

$$\operatorname{atan}(y, x) = \begin{cases} \operatorname{atan}(y/x) & \text{quadrants I and IV} \\ -\pi + \operatorname{atan}(y/x) & \text{quadrant III} \\ \pi + \operatorname{atan}(y/x) & \text{quadrant II} \end{cases} \tag{10}$$

In Figure 3, we indicate the range of $\theta = \operatorname{atan}(y, x)$ for each quadrant.

C. Euler's formula

The polar form is often convenient because of an extremely important identity, known as *Euler's formula*:

$$e^{\mathbf{j}\theta} = \cos\theta + \mathbf{j}\sin\theta \tag{11}$$

Equation (11) allows us to express any complex number as a *complex exponential*:

$$z = x + \mathbf{j}y = r\cos\theta + \mathbf{j}r\sin\theta = re^{\mathbf{j}\theta}. \tag{12}$$

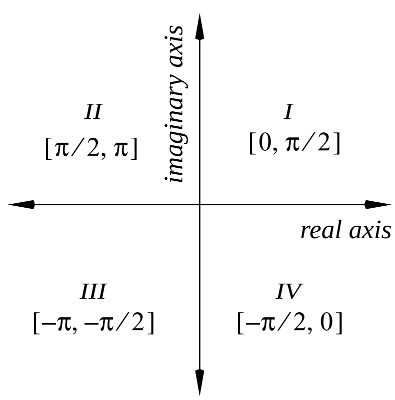

Figure 3

for which all the standard laws of exponents apply. As such, it will help us deal with sinusoidal functions using exponentials, for which the algebra is typically much more straightforward.

Given the polar form of complex numbers, we define two more quantities related to complex numbers:

$$|z| = r = \sqrt{x^2 + y^2} = \textit{magnitude} \text{ of the complex number } z\text{, and,} \tag{13}$$

$$\arg(z) = \theta = \operatorname{atan}(y, x) = \textit{phase} \text{ of the complex number } z \text{ (in radians).}^1 \tag{14}$$

Proof of Euler's formula: We can show that Euler's formula makes sense by expanding each term in equation (11) as its Taylor Series expansion. Recall from calculus that any function in x can be approximated by a polynomial about $x = 0$. The relevant Taylor series expansions are given below:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{15}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \tag{16}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{k=1}^{\infty} \frac{(-1)^k x^{(2k+1)}}{(2k+1)!} \tag{17}$$

where,

$$n! = n \times (n-1) \times \dots \times 2 \times 1\,. \tag{18}$$

Substituting equations (15) through (17) into equation (11), we get:

$$1 + \mathbf{j}\theta + \frac{(\mathbf{j}\theta)^2}{2!} + \frac{(\mathbf{j}\theta)^3}{3!} + \frac{(\mathbf{j}\theta)^4}{4!} + \frac{(\mathbf{j}\theta)^5}{5!} + \dots = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + \mathbf{j}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right) \tag{19}$$

where, for convenience, we have truncated each Taylor series up to the fifth-order term. Using,

$$\mathbf{j}^2 = -1\,, \tag{20}$$

we can simplify equation (19) as:

$$1 + \mathbf{j}\theta - \frac{\theta^2}{2!} - \frac{\mathbf{j}(\theta)^3}{3!} + \frac{\theta^4}{4!} + \frac{\mathbf{j}(\theta)^5}{5!} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\right) + \mathbf{j}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right) \tag{21}$$

We can readily observe that equation (21) is indeed correct by verifying that every term that appears on the left-hand side of the equation also appears on the right-hand side of the equation. Consequently, Euler's formula must be correct.

D. Inverse Euler relations

Euler's formula in equation (11) can be inverted to give the following extremely important *inverse Euler relations*:

$$\cos\theta = \frac{e^{\mathbf{j}\theta} + e^{-\mathbf{j}\theta}}{2} \tag{22}$$

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1. *So far in the class, I have frequently referred to the "magnitude spectrum" of a signal. As we will see soon, the frequency representation of a time-domain signal is, in general, a complex-valued function for which we typically plot the magnitude and phase separately as a function of frequency. Once we define the frequency transform formally, we will be concerned not only with the "magnitude spectrum", but "phase spectrum" as well.*

$$\sin\theta = \frac{e^{\mathbf{j}\theta} - e^{-\mathbf{j}\theta}}{2\mathbf{j}} \tag{23}$$

This can be shown relatively easily. Let us use Euler's formula to expand $e^{-\mathbf{j}\theta}$:

$$e^{-\mathbf{j}\theta} = \cos(-\theta) + \mathbf{j}\sin(-\theta) \tag{24}$$

Note that the cosine function is an *even* function, while the sine function is an *odd* function, so that:

$$\cos(-\theta) = \cos(\theta) \tag{25}$$

$$\sin(-\theta) = -\sin(\theta)\,. \tag{26}$$

Therefore equation (24) may be simplified to:

$$e^{-\mathbf{j}\theta} = \cos\theta - \mathbf{j}\sin\theta\,. \tag{27}$$

Using equations (11) and (24), we can now show that the left-hand and right-hand sides of equation (22) are equal:

$$\begin{aligned}\cos\theta &= \frac{e^{\mathbf{j}\theta} + e^{-\mathbf{j}\theta}}{2} \\ &= \frac{(\cos\theta + \mathbf{j}\sin\theta) + (\cos\theta - \mathbf{j}\sin\theta)}{2} \\ &= \frac{2\cos\theta}{2} \\ &= \cos\theta\end{aligned} \tag{28}$$

The same can be done for equation (23):

$$\begin{aligned}\sin\theta &= \frac{e^{\mathbf{j}\theta} - e^{-\mathbf{j}\theta}}{2\mathbf{j}} \\ &= \frac{(\cos\theta + \mathbf{j}\sin\theta) - (\cos\theta - \mathbf{j}\sin\theta)}{2\mathbf{j}} \\ &= \frac{\mathbf{j}2\sin\theta}{2\mathbf{j}} \\ &= \sin\theta\end{aligned} \tag{29}$$

The inverse Euler formulas go a long way towards explaining the frequency representation of a sinusoidal signal. Let us consider the following function:

$$x(t) = A\cos(2\pi f_o t) \tag{30}$$

Using equation (22), we can rewrite equation (30) as:

$$x(t) = \frac{1}{2}Ae^{\mathbf{j}2\pi f_o t} + \frac{1}{2}Ae^{-\mathbf{j}2\pi f_0 t} \tag{31}$$

Note that the complex representation in equation (31) has two components, each 1/2 the magnitude of the signal $x(t)$, with a positive and negative exponential, corresponding to the negative and positive frequency terms with which we have seen many times in the spectrum representation $X(f)$ of a sinusoid (see Figure 4).

While equation (31) still does not fully explain the spectrum representation $X(f)$ as two shifted Dirac delta functions, it does take us one step closer.

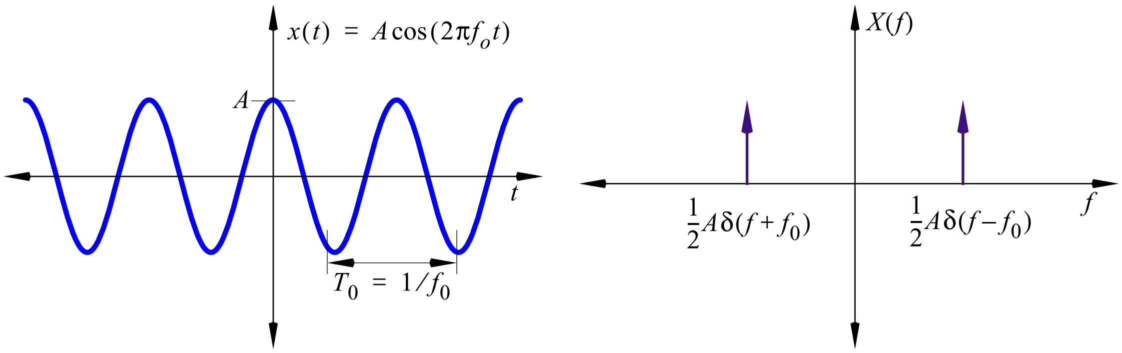


Figure 4

3. Complex algebra

A. Introduction

Below we derive basic algebraic operations on complex numbers. As we will see, some of these operations are easier to carry out in the Cartesian form,

$$z = x + \mathbf{j}y \tag{32}$$

while others are easier in the polar form,

$$z = re^{\mathbf{j}\theta}. \tag{33}$$

In our derivations below let,

$$z_1 = x_1 + \mathbf{j}y_1 = r_1 e^{\mathbf{j}\theta_1} \tag{34}$$

$$z_2 = x_2 + \mathbf{j}y_2 = r_2 e^{\mathbf{j}\theta_2}, \tag{35}$$

where,

$$r_1 = \sqrt{x_1^2 + y_1^2},\ r_2 = \sqrt{x_2^2 + y_2^2},\ \theta_1 = \mathrm{atan}(y_1, x_1) \text{ and } \theta_2 = \mathrm{atan}(y_2, x_2), \tag{36}$$

and, conversely,

$$x_1 = r_1\cos(\theta_1),\ y_1 = r_1\sin(\theta_1),\ x_2 = r_2\cos(\theta_2) \text{ and } y_2 = r_2\sin(\theta_2). \tag{37}$$

B. Addition

In Cartesian form, addition of two complex numbers is relatively straightforward:

$$\begin{aligned} z_1 + z_2 &= (x_1 + \mathbf{j}y_1) + (x_2 + \mathbf{j}y_2) \\ &= (x_1 + x_2) + \mathbf{j}(y_1 + y_2) \end{aligned} \tag{38}$$

If z_1 and z_2 are given in polar form, one must first convert the two numbers into Cartesian form, and then follow equation (38) above:

$$\begin{aligned} z_1 + z_2 &= r_1 e^{\mathbf{j}\theta_1} + r_2 e^{\mathbf{j}\theta_2} \\ &= [r_1\cos(\theta_1) + r_2\cos(\theta_2)] + \mathbf{j}[r_1\sin(\theta_1) + r_2\sin(\theta_2)] \end{aligned}. \tag{39}$$

The result in equation (39) can then be converted back to polar form:

$$x_3 = r_1\cos(\theta_1) + r_2\cos(\theta_2) \tag{40}$$

$$y_3 = r_1 \sin(\theta_1) + r_2 \sin(\theta_2) \tag{41}$$

$$r_3 = \sqrt{x_3^2 + y_3^2} \tag{42}$$

$$\theta_3 = \operatorname{atan}(y_3, z_3) \tag{43}$$

$$z_1 + z_2 = r_3 e^{\mathbf{j}\theta_3}. \tag{44}$$

C. Multiplications

In Cartesian form, multiplication of two complex numbers is given by,

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + \mathbf{j}y_1) \cdot (x_2 + \mathbf{j}y_2) \\ &= x_1 x_2 + \mathbf{j}x_1 y_2 + \mathbf{j}x_2 y_1 + \mathbf{j}^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + \mathbf{j}(x_1 y_2 + x_2 y_1) \end{aligned} \tag{45}$$

Multiplication is easier to carry out in polar form:

$$\begin{aligned} z_1 \cdot z_2 &= r_1 e^{\mathbf{j}\theta_1} \cdot r_2 e^{\mathbf{j}\theta_2} \\ &= r_1 r_2 e^{\mathbf{j}(\theta_1 + \theta_2)} \end{aligned} \tag{46}$$

D. Division

In Cartesian form, division of two complex numbers is given by,

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{(x_1 + \mathbf{j}y_1)}{(x_2 + \mathbf{j}y_2)} \\ &= \frac{(x_1 + \mathbf{j}y_1)}{(x_2 + \mathbf{j}y_2)} \cdot \frac{(x_2 - \mathbf{j}y_2)}{(x_2 - \mathbf{j}y_2)} \\ &= \frac{(x_1 x_2 + y_1 y_2) + \mathbf{j}(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} \end{aligned} \tag{47}$$

Division is easier to carry out in polar form:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1 e^{\mathbf{j}\theta_1}}{r_2 e^{\mathbf{j}\theta_2}} \\ &= \frac{r_1}{r_2} e^{\mathbf{j}(\theta_1 - \theta_2)} \end{aligned} \tag{48}$$

E. Exponentiation

Below, we compute z^a, where,

$$z = x + \mathbf{j}y = re^{\mathbf{j}\theta} \tag{49}$$

In polar form, z^a is relatively straightforward to compute:

$$\begin{aligned} z^a &= (re^{\mathbf{j}\theta})^a \\ &= r^a e^{\mathbf{j}a\theta} \end{aligned} \tag{50}$$

In Cartesian form, it is easiest to first convert the number to polar form, use equation (50) above, and then convert back to Cartesian form:

$$\begin{aligned} z^a &= (x + \mathbf{j}y)^a \\ &= (\sqrt{x^2+y^2})^a e^{\mathbf{j}[a \cdot \operatorname{atan}(y,x)]} \\ &= (x^2+y^2)^{a/2} e^{\mathbf{j}[a \cdot \operatorname{atan}(y,x)]} \end{aligned} \tag{51}$$

If we let,

$$r' = (x^2+y^2)^{a/2} \tag{52}$$

$$\theta' = a \cdot \operatorname{atan}(y,x) \tag{53}$$

then, the result in equation (51) can be written in Cartesian form as:

$$z^a = r'\cos(\theta') + \mathbf{j}r'\sin(\theta')\,. \tag{54}$$

F. Conjugate

The conjugate z^* of a complex number,

$$z = x + \mathbf{j}y = re^{\mathbf{j}\theta} \tag{55}$$

is given by,

$$z^* = x - \mathbf{j}y \text{ in Cartesian form and,} \tag{56}$$

$$z^* = re^{-\mathbf{j}\theta} \text{ in polar form.} \tag{57}$$

4. Numeric examples

See the web site for the *Mathematica* notebook “complex_examples.nb” and/or *Matlab* session log “complex_examples.m” for all of the numeric examples below.

A. Conversion between Cartesian and polar forms

Below, we give some examples of complex number conversions between Cartesian and polar forms:

$$e^{\mathbf{j}(\pi/2)} = \cos(\pi/2) + \mathbf{j}\sin(\pi/2) = \mathbf{j} \tag{58}$$

$$e^{\mathbf{j}\pi} = \cos(\pi) + \mathbf{j}\sin(\pi) = -1 \tag{59}$$

$$e^{-\mathbf{j}(\pi/2)} = \cos(-\pi/2) + \mathbf{j}\sin(-\pi/2) = -\mathbf{j} \tag{60}$$

$$e^{\mathbf{j}n\pi} = \cos(n\pi) + \mathbf{j}\sin(n\pi) = \begin{cases} 1 & n \in \{0, \pm 2, \pm 4, \ldots\} \\ -1 & n \in \{\pm 1, \pm 3, \pm 5, \ldots\} \end{cases} \tag{61}$$

$$\sqrt{2}e^{\mathbf{j}(3\pi/4)} = \sqrt{2}\cos(3\pi/4) + \mathbf{j}\sqrt{2}\sin(3\pi/4) = \sqrt{2}\cdot\left(-\frac{1}{\sqrt{2}}\right) + \mathbf{j}\sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}\right) = -1 + \mathbf{j} \tag{62}$$

$$0 + \mathbf{j}2 = \sqrt{0^2+2^2}e^{\mathbf{j}\operatorname{atan}(2,0)} = 2e^{\mathbf{j}(\pi/2)} \tag{63}$$

$$-1 + \mathbf{j} = \sqrt{(-1)^2+1^2}e^{\mathbf{j}\operatorname{atan}(1,-1)} = \sqrt{2}e^{\mathbf{j}(3\pi/4)} \tag{64}$$

$$-3 - \mathbf{j}4 = \sqrt{(-3)^2+(-4)^2}e^{\mathbf{j}\operatorname{atan}(-4,-3)} \approx 5e^{-\mathbf{j}2.2143} \tag{65}$$

B. Algebraic operations

Below we give some examples of algebraic operations with complex numbers:

$$\mathbf{j}^3 = [e^{\mathbf{j}(\pi/2)}]^3 = e^{\mathbf{j}(3\pi/2)} = -\mathbf{j} \tag{66}$$

$$\begin{aligned} 3e^{\mathbf{j}(2\pi/3)} - 4e^{-\mathbf{j}(\pi/6)} &= [3\cos(2\pi/3) + \mathbf{j}3\sin(2\pi/3)] + [-4\cos(-\pi/6) - \mathbf{j}4\sin(-\pi/6)] \\ &= 3\left(-\frac{1}{2}\right) + \mathbf{j}3\left(\frac{\sqrt{3}}{2}\right) - 4\left(\frac{\sqrt{3}}{2}\right) - \mathbf{j}4\left(-\frac{1}{2}\right) \\ &= \left(\frac{-3}{2} - 2\sqrt{3}\right) + \mathbf{j}\left(\frac{3\sqrt{3}}{2} + 2\right) \end{aligned} \tag{67}$$

$$\begin{aligned} (\sqrt{2} - \mathbf{j}2)^8 &= [\sqrt{(\sqrt{2})^2 + (-2)^2}e^{\mathbf{j}\,\mathrm{atan}(-2,\,\sqrt{2})}]^8 \\ &= [\sqrt{6}e^{-\mathbf{j}0.955317}]^8 \\ &= (\sqrt{6})^8 e^{-\mathbf{j}7.64253} \\ &= 1296e^{-\mathbf{j}7.64253} \\ &= 1296\cos(-7.64253) + \mathbf{j}1296\sin(-7.64253) \\ &= 272 - \mathbf{j}1267.14 \end{aligned} \tag{68}$$

$$(\sqrt{2} - \mathbf{j}2)^{-1} = \frac{1}{\sqrt{2} - \mathbf{j}2} = \frac{1}{\sqrt{2} - \mathbf{j}2} \cdot \frac{\sqrt{2} + \mathbf{j}2}{\sqrt{2} + \mathbf{j}2} = \frac{\sqrt{2} + \mathbf{j}2}{6} \approx 0.235702 + \mathbf{j}0.333333 \tag{69}$$

$$\begin{aligned} (\sqrt{2} - \mathbf{j}2)^{-1} &= [\sqrt{6}e^{\mathbf{j}\,\mathrm{atan}(-2,\,\sqrt{2})}]^{-1} \\ &= [\sqrt{6}e^{-\mathbf{j}0.955317}]^{-1} \\ &= \left(\frac{1}{\sqrt{6}}\right)e^{-\mathbf{j}0.955317} \\ &= \frac{1}{\sqrt{6}}\cos(-0.955317) + \mathbf{j}\frac{1}{\sqrt{6}}\sin(-0.955317) \\ &= 0.235702 + \mathbf{j}0.333333 \end{aligned} \tag{70}$$

[Note that equations (69) and (70) yield the same result, through two different approaches.]

$$\begin{aligned} Im[\mathbf{j}e^{-\mathbf{j}(\pi/3)}] &= Im[\mathbf{j}(\cos(-\pi/3) + \mathbf{j}\sin(-\pi/3))] \\ &= Im\left[\mathbf{j}\left(\frac{1}{2} + \mathbf{j}\left(\frac{-\sqrt{3}}{2}\right)\right)\right] \\ &= Im\left[\frac{\mathbf{j}}{2} + \frac{\sqrt{3}}{2}\right] \\ &= 1/2 \end{aligned} \tag{71}$$

For the derivations below, assume the following complex numbers:

$$z_1 = -4 + \mathbf{j}3 \text{ and } z_2 = 1 - \mathbf{j}. \tag{72}$$

Note that the corresponding polar forms are given by,

$$z_1 = 5e^{\mathbf{j}\,\mathrm{atan}(3,\,-4)} \text{ and } z_2 = \sqrt{2}e^{-\mathbf{j}(\pi/4)}. \tag{73}$$

Below, some of the operations are carried out in either Cartesian or polar form, while others are carried out using both forms, to show that the same outcome results no matter which form is used.

$$z_1{}^* = -4 - \mathbf{j}3 \tag{74}$$

$$z_2^2 = (1 - \mathbf{j})^2 = (1 - \mathbf{j})(1 - \mathbf{j}) = 1 - \mathbf{j}2 + \mathbf{j}^2 = -\mathbf{j}2 \tag{75}$$

$$z_2^2 = [\sqrt{2}e^{-\mathbf{j}(\pi/4)}]^2 = 2e^{-\mathbf{j}(\pi/2)} = -\mathbf{j}2 \tag{76}$$

$$z_1 + z_2{}^* = (-4 + \mathbf{j}3) + (1 + \mathbf{j}) = -3 + \mathbf{j}4 \tag{77}$$

$$\mathbf{j}z_2 = \mathbf{j}(1 - \mathbf{j}) = 1 + \mathbf{j} \tag{78}$$

$$\mathbf{j}z_2 = e^{\mathbf{j}(\pi/2)}\sqrt{2}e^{-\mathbf{j}(\pi/4)} = \sqrt{2}e^{\mathbf{j}(\pi/4)} = \sqrt{2}\cos(\pi/4) + \mathbf{j}\sqrt{2}\sin(\pi/4) = 1 + \mathbf{j} \tag{79}$$

$$z_1 z_1{}^* = (-4 + \mathbf{j}3)(-4 - \mathbf{j}3) = 16 - \mathbf{j}^2 9 = 25 \tag{80}$$

$$z_1 z_1{}^* = 5e^{\mathbf{j}\,\mathrm{atan}(3,-4)} \cdot 5e^{-\mathbf{j}\,\mathrm{atan}(3,-4)} = 25 \tag{81}$$

$$e^{z_2} = e^{(1-\mathbf{j})} = e \cdot e^{-\mathbf{j}} = e\cos(-1) + \mathbf{j}e\sin(-1) \approx 1.46869 - \mathbf{j}2.28736 \tag{82}$$

$$z_1/z_2 = \frac{-4 + \mathbf{j}3}{1 - \mathbf{j}} = \frac{-4 + \mathbf{j}3}{1 - \mathbf{j}} \cdot \frac{1 + \mathbf{j}}{1 + \mathbf{j}} = \frac{-7 - \mathbf{j}}{2} = -3.5 - \mathbf{j}0.5 \tag{83}$$

$$\begin{aligned} z_1/z_2 &= \frac{5e^{\mathbf{j}\,\mathrm{atan}(3,-4)}}{\sqrt{2}e^{-\mathbf{j}(\pi/4)}} = \frac{5}{\sqrt{2}}e^{\mathbf{j}[\mathrm{atan}(3,-4) + \pi/4]} \approx 3.53553e^{\mathbf{j}3.28349} \\ &= 3.53553\cos(3.28349) + \mathbf{j}3.53553\sin(3.28349) \\ &= -3.5 - \mathbf{j}0.5 \end{aligned} \tag{84}$$

5. Trigonometric analysis using complex variables

The principal reason we introduce complex variables in our study of signals and systems is that they significantly simply mathematical analysis, converting trigonometric functions to complex exponentials through Euler's formula. Below, we show how relatively straightforward it is to verify trigonometric identities using complex exponentials and the Euler relationship.

We will specifically make use of the inverse Euler relations,

$$\cos\theta = \frac{e^{\mathbf{j}\theta} + e^{-\mathbf{j}\theta}}{2} \tag{85}$$

$$\sin\theta = \frac{e^{\mathbf{j}\theta} - e^{-\mathbf{j}\theta}}{2\mathbf{j}} \tag{86}$$

presented previously in class and derived from Euler's equation:

$$e^{\mathbf{j}\theta} = \cos\theta + \mathbf{j}\sin\theta\,. \tag{87}$$

A. Example #1

Below, we prove the trigonometric identity,

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{88}$$

using complex exponentials. Substituting equations (85) and (86) into (88):

$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{e^{\mathbf{j}\theta}-e^{-\mathbf{j}\theta}}{2\mathbf{j}}\right)\left(\frac{e^{\mathbf{j}\theta}+e^{-\mathbf{j}\theta}}{2}\right) \\ &= \frac{2}{4\mathbf{j}}(e^{\mathbf{j}2\theta}-e^{-\mathbf{j}2\theta}) \\ &= \frac{e^{\mathbf{j}2\theta}-e^{-\mathbf{j}2\theta}}{2\mathbf{j}} \\ &= \sin(2\theta)\end{aligned} \tag{89}$$

B. Example #2

Below, we prove the trigonometric identity,

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{90}$$

using complex exponentials. Substituting equations (85) and (86) into (90):

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\frac{e^{\mathbf{j}\alpha}+e^{-\mathbf{j}\alpha}}{2}\right)\left(\frac{e^{\mathbf{j}\beta}+e^{-\mathbf{j}\beta}}{2}\right) - \left(\frac{e^{\mathbf{j}\alpha}-e^{-\mathbf{j}\alpha}}{2\mathbf{j}}\right)\left(\frac{e^{\mathbf{j}\beta}-e^{-\mathbf{j}\beta}}{2\mathbf{j}}\right) \\ &= \frac{1}{4}(e^{\mathbf{j}(\alpha+\beta)}+e^{\mathbf{j}(\alpha-\beta)}+e^{\mathbf{j}(-\alpha+\beta)}+e^{\mathbf{j}(-\alpha-\beta)}) - \\ &\quad \frac{1}{4\mathbf{j}^2}(e^{\mathbf{j}(\alpha+\beta)}-e^{\mathbf{j}(\alpha-\beta)}-e^{\mathbf{j}(-\alpha+\beta)}+e^{\mathbf{j}(-\alpha-\beta)})\end{aligned} \tag{91}$$

Since,

$$\frac{1}{4\mathbf{j}^2} = -\frac{1}{4} \tag{92}$$

we can combine terms from the first and second terms in (91):

$$\begin{aligned}\cos\alpha\cos\beta - \sin\alpha\sin\beta &= \frac{1}{4}(e^{\mathbf{j}(\alpha+\beta)}+e^{\mathbf{j}(\alpha-\beta)}+e^{\mathbf{j}(-\alpha+\beta)}+e^{\mathbf{j}(-\alpha-\beta)}) + \\ &\quad \frac{1}{4}(e^{\mathbf{j}(\alpha+\beta)}-e^{\mathbf{j}(\alpha-\beta)}-e^{\mathbf{j}(-\alpha+\beta)}+e^{\mathbf{j}(-\alpha-\beta)}) \\ &= \frac{1}{4}(2e^{\mathbf{j}(\alpha+\beta)}+2e^{\mathbf{j}(-\alpha-\beta)}) \\ &= \frac{e^{\mathbf{j}(\alpha+\beta)}+e^{-\mathbf{j}(\alpha+\beta)}}{2} \\ &= \cos(\alpha+\beta)\end{aligned} \tag{93}$$

Using similar proofs, it is easy to show the following important trigonometric identities:

$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{94}$$

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta\,. \tag{95}$$

C. Example #3: DeMoivre’s formula

Below, we prove DeMoivre’s formula,

$$(\cos\theta+\mathbf{j}\sin\theta)^n = \cos(n\theta)+\mathbf{j}\sin(n\theta) \tag{96}$$

using complex exponentials. Using Euler’s equation,

$$\begin{aligned}(\cos\theta + \mathbf{j}\sin\theta)^n &= \cos(n\theta) + \mathbf{j}\sin(n\theta)\\ (e^{\mathbf{j}\theta})^n &= e^{\mathbf{j}n\theta}\\ e^{\mathbf{j}n\theta} &= e^{\mathbf{j}n\theta}\end{aligned} \tag{97}$$

6. Plotting complex functions

Here, we discuss the problem of plotting functions of complex variables $f(z)$. The basic problem in visualizing functions of complex variables is that there are now two dimensions for the independent variable z — namely, $Re[z]$ and $Im[z]$, *or* $|z|$ and $\arg(z)$, and two dimensions for the function value $f(z)$ — namely, $Re[f(z)]$ and $Im[f(z)]$, *or* $|f(z)|$ and $\arg(f(z))$. Therefore, we need four dimensions to plot functions of complex variables. One way to get around this problem is to show how the function transforms sets of lines that lie in the complex plane. Each line in the input space (complex plane) will be mapped into some curve in the output space (complex plane), each of which can be represented in two dimensions. This method of visualizing complex functions is known as *complex mapping*.

A. Complex mappings

Figures 5 through 8 show some complex mappings for the following simple functions:

$$f(z) = e^z \text{ (Figure 5)} \tag{98}$$

$$f(z) = \sqrt{z} \text{ (Figure 6)} \tag{99}$$

$$f(z) = \sin(z) \text{ (Figures 7 and 8)} \tag{100}$$

For Figures 5 and 7, the input space lines in the complex plane are defined as a rectangular grid, while for Figures 7 and 8, the input space lines are defined as a polar grid. The rectangular gird corresponds to the Cartesian form $z = x + \mathbf{j}y$, while the polar grid corresponds to the polar form $z = re^{\mathbf{j}\theta}$. While most of these mappings are not particularly intuitive (that is, it is difficult to derive these mappings by hand), the mapping in Figure 6 is relatively straightforward to justify. For this polar mapping,

$$f(re^{\mathbf{j}\theta}) = \sqrt{re^{\mathbf{j}\theta}} = \sqrt{r}e^{\mathbf{j}(\theta/2)} \tag{101}$$

That is, the square root function halves the input angle, as is shown in Figure 6.

B. One-dimensional complex plots

Fortunately, in this class, we will most often be concerned with complex functions where we vary only one of the dimensions in the input space — namely, $\arg(z) = \theta$, so that the input space is now one-dimensional. Thus, rather than resort to complex mappings, we can plot $f(e^{\mathbf{j}\theta})$ as two separate plots:

$$|f(e^{\mathbf{j}\theta})| \text{ and,} \tag{102}$$

$$\arg(f(e^{\mathbf{j}\theta})).^{1} \tag{103}$$

Term (102) above corresponds to the *magnitude* plot of f as a function of θ, while term (103) corresponds to the *phase* plot of f as a function of θ. Figures 9 through 11 below plot $|f(z)|$ and $\arg(f(z))$ for $z = e^{\mathbf{j}\theta}$ and the following example complex-valued functions:

$$f(z) = e^z \text{ (Figure 9)} \tag{104}$$

$$f(z) = \sqrt{z} \text{ (Figure 10)} \tag{105}$$

$$f(z) = \sin(z) \text{ (Figure 11).} \tag{106}$$

(Figures 5 through 11 were generated with the *Mathematica* notebook “complex_functions.nb”.)

1. *While we could also plot $Re[f(e^{\mathbf{j}\theta})]$ and $Im[f(e^{\mathbf{j}\theta})]$ as functions of θ, we will find that the magnitude and phase functions in (102) and (103) above will offer more physical insight.*

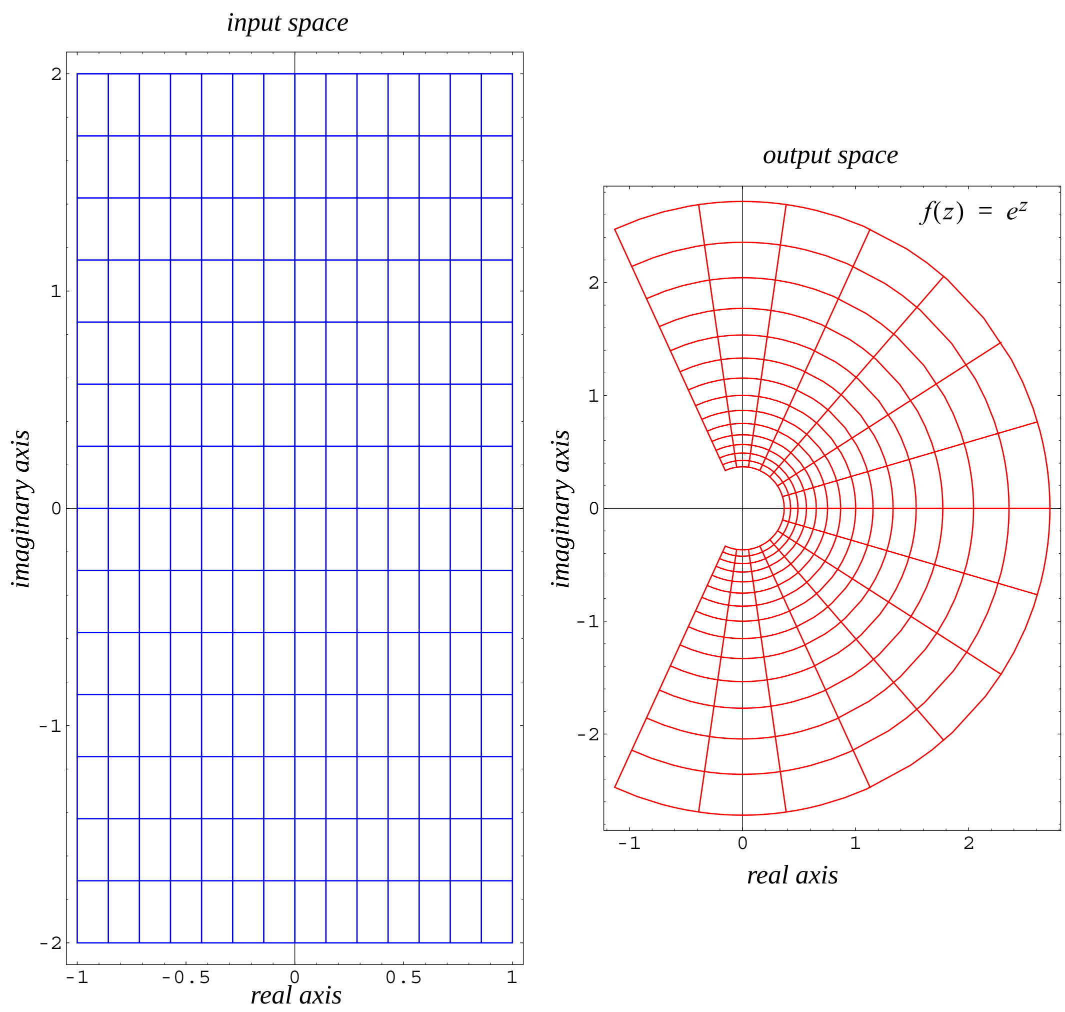


Figure 5

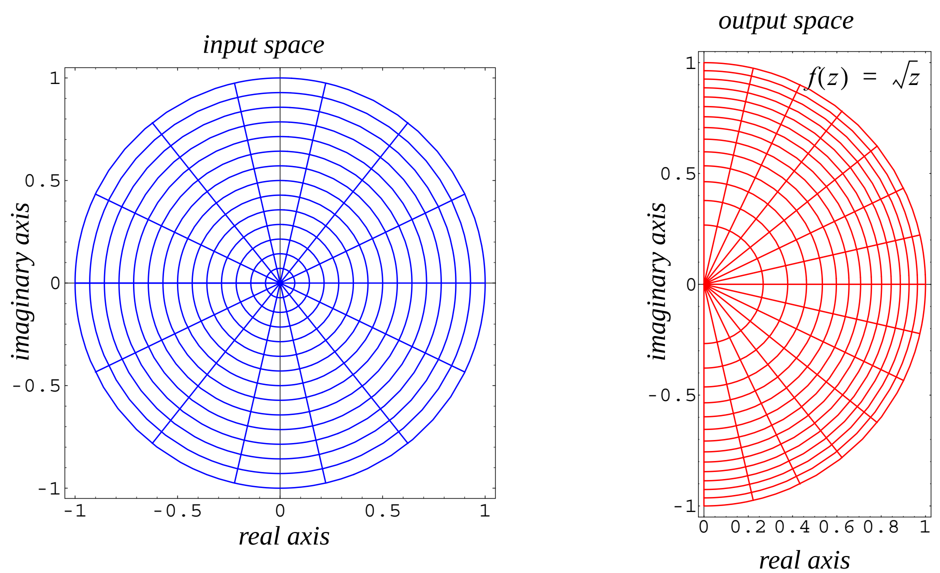


Figure 6

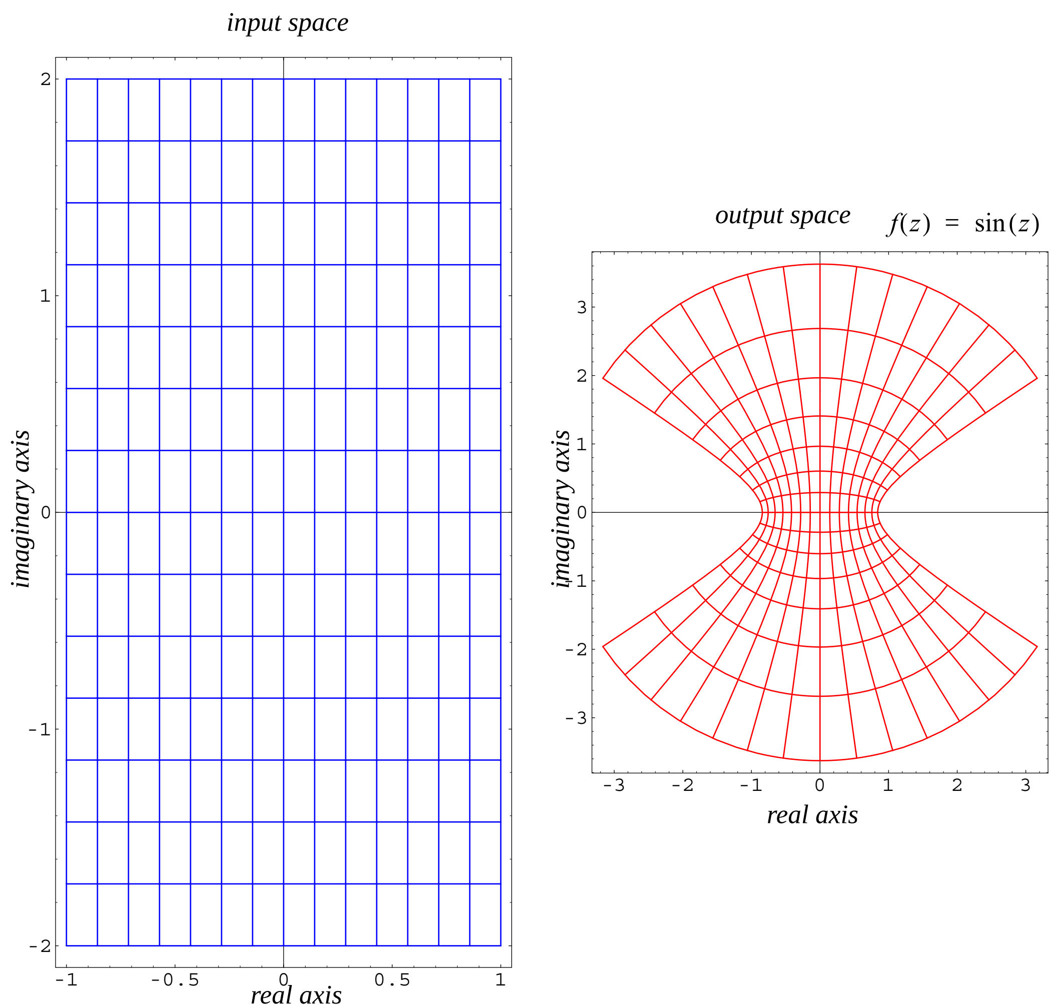


Figure 7

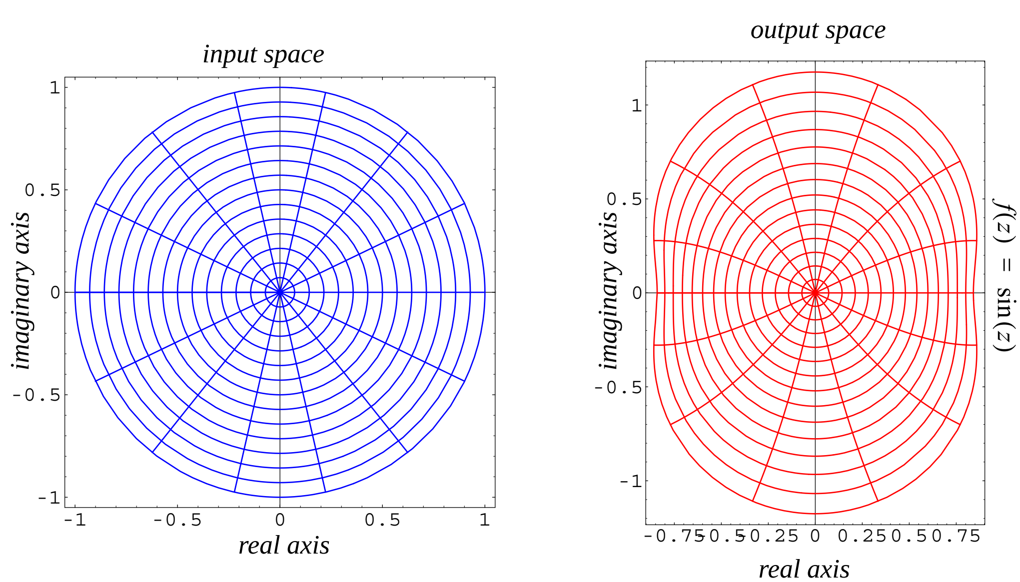
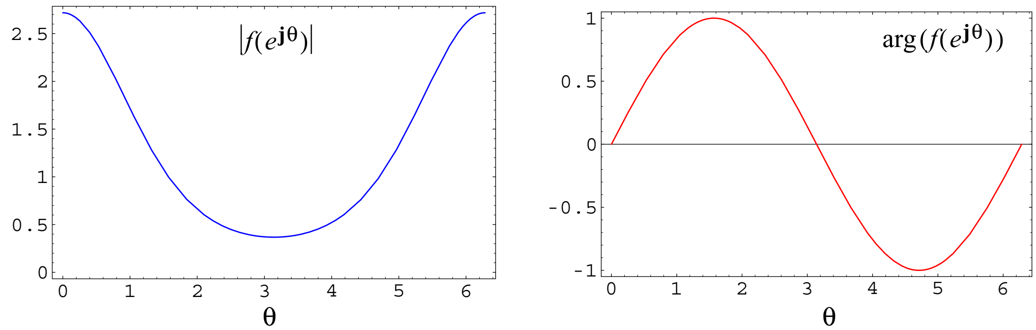
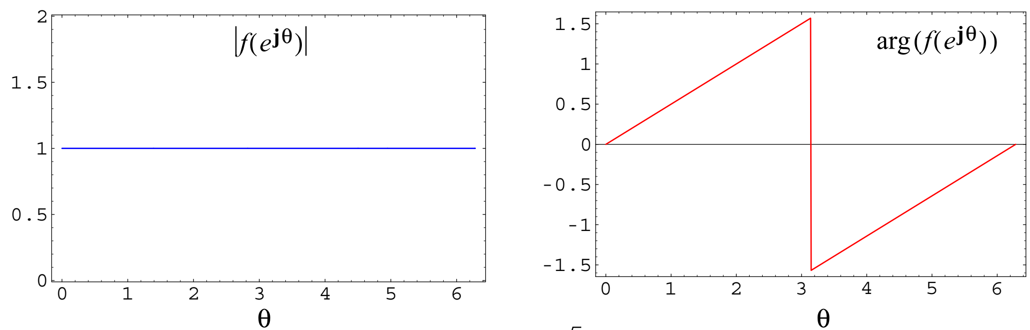


Figure 8



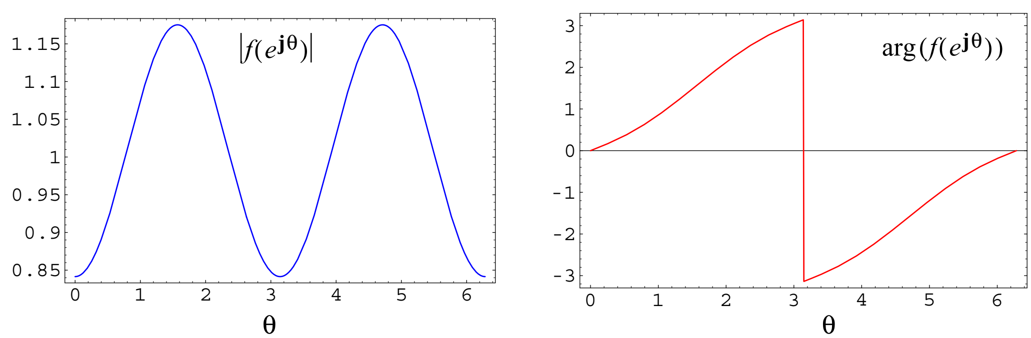
$f(z) = e^z$

Figure 9



$f(z) = \sqrt{z}$

Figure 10



$f(z) = \sin(z)$

Figure 11