Introduction to Fourier analysis

1. Introduction

In this lecture, we introduced the basic transforms that allow us to analyze the frequency content of continuous-time and discrete-time signals.

2. Introduction to Fourier analysis

As we have already seen in this course, being able to analyze the frequency content of a time-domain signal is critically important to understanding and manipulating signals in desirable ways, such as filtering. *Fourier analysis* refers to a set of techniques that allow us to do just that. Depending on the type of signal, we will see that different, though related *transforms* are applicable. The table below divides signals by two criteria: (1) periodic *vs.* nonperiodic and (2) continuous-time *vs.* discrete-time.[1] For each box in the table, we specify which transforms are applicable, and their respective definitions.

	Continuous-Time	*Discrete-Time*
Periodic	***Fourier Series (FS):*** $X_k = \frac{1}{T_0}\left[\int_{t_0}^{(t_0+T_0)} x(t)e^{-\mathbf{j}2\pi k f_0 t}dt\right]$ $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathbf{j}2\pi k f_0 t}$ $x(t) = X_0 + 2\sum_{k=1}^{\infty} \lvert X_k\rvert \cos(2\pi k f_0 t + \angle X_k)$	***Discrete-Time Fourier Transform (DTFT):*** $X(e^{\mathbf{j}\theta}) = \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}n\theta}$ ***Inverse Discrete-Time Fourier Transform (IDTFT):*** $x[n] = \frac{1}{2\pi}\int_{\theta_0}^{(\theta_0+2\pi)} X(e^{\mathbf{j}\theta})e^{\mathbf{j}n\theta}d\theta$
Non-periodic	***Fourier Transform (FT):*** $X(f) = \int_{-\infty}^{\infty} x(t)e^{-\mathbf{j}2\pi f t}dt$ ***Inverse Fourier Transform (IFT):*** $x(t) = \int_{-\infty}^{\infty} X(f)e^{\mathbf{j}2\pi f t}df$	***Discrete Fourier Transform (DFT, FFT):*** $X(k) = \sum_{n=0}^{N-1} x[n]e^{-\mathbf{j}2\pi nk/N}$ ***Inverse Discrete Fourier Transform (IDFT, IFFT):*** $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{\mathbf{j}2\pi nk/N}$

A. Continuous-time Fourier series

For continuous-time signals, the *Fourier series* allows us to represent almost any periodic signal $x(t)$ as an infinite sum of *harmonically* related complex exponentials or sinusoids:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathbf{j}2\pi k f_0 t} \tag{1}$$

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} \lvert X_k\rvert \cos(2\pi k f_0 t + \angle X_k) \tag{2}$$

where,

1. *We do not make a distinction between periodic and non-periodic signals for discrete-time signals, as such a distinction at this point would not be meaningful.*

$$X_k = \frac{1}{T_0}\left[\int_{t_0}^{(t_0+T_0)} x(t)e^{-\mathbf{j}2\pi k f_0 t}dt\right] = k\text{th Fourier coefficient,} \tag{3}$$

$$T_0 = \text{period of } x(t) \text{ (fundamental period), and,} \tag{4}$$

$$f_0 = 1/T_0 = \text{fundamental frequency of } x(t)\,. \tag{5}$$

Equation (1) represents the complex exponential form of the Fourier series, while equation (2) represents the sinusoidal form of the Fourier series for periodic signal $x(t)$. Another way to look at the Fourier series decomposition of a periodic signal is in the frequency domain, where the coefficient X_k corresponds to frequency $f = kf_0$. In general, X_k may be a complex number. That is, a continuous-time, periodic signal consists of a discrete number of non-zero frequency components X_k at:

$$f = kf_0\,,\ k \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}\,, \tag{6}$$

with magnitude $|X_k|$ and phase $\arg(X_k)$. The frequencies f in equation (6) are said to be *harmonics* of the fundamental frequency f_0.

B. Continuous-time Fourier transform

For nonperiodic, continuous-time signals, the *Fourier transform* allows us to determine the frequency content of a signal $x(t)$:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-\mathbf{j}2\pi f t}dt \tag{7}$$

$X(f)$ is a complex function of the frequency variable f, and tells us the frequency composition of $x(t)$ at every frequency f. We can completely recover the original signal $x(t)$ from the Fourier transform $X(f)$ using the *inverse Fourier transform*:

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{\mathbf{j}2\pi f t}df \tag{8}$$

Recall that previously we claimed that the frequency representation of the cosine function,

$$x(t) = A\cos(2\pi f_0 t) \tag{9}$$

is given by,

$$X(f) = \frac{A}{2}\delta(f+f_0) + \frac{A}{2}\delta(f-f_0)\,. \tag{10}$$

We are now in position to verify this frequency representation of the cosine function, using the inverse Fourier transform. Plugging (10) into equation (8) and using the sifting property of the δ function,

$$x(t) = \int_{-\infty}^{\infty}\left[\frac{A}{2}\delta(f+f_0) + \frac{A}{2}\delta(f-f_0)\right]e^{\mathbf{j}2\pi f t}df \tag{11}$$

$$x(t) = \int_{-\infty}^{\infty}\left(\frac{A}{2}\delta(f+f_0)e^{\mathbf{j}2\pi f t} + \frac{A}{2}\delta(f-f_0)e^{\mathbf{j}2\pi f t}\right)df \tag{12}$$

$$x(t) = \frac{A}{2}e^{-\mathbf{j}2\pi f_0 t} + \frac{A}{2}e^{\mathbf{j}2\pi f_0 t} \tag{13}$$

$$x(t) = A\left(\frac{e^{-\mathbf{j}2\pi f_0 t} + e^{\mathbf{j}2\pi f_0 t}}{2}\right) \tag{14}$$

$$x(t) = A\cos(2\pi f_0 t) \tag{15}$$

In general, for a shifted cosine function,

$$x(t) = A\cos(2\pi f_0 t + \alpha) \tag{16}$$

the Fourier transform is given by,

$$X(f) = \frac{A}{2}e^{-\mathbf{j}\alpha}\delta(f+f_0) + \frac{A}{2}e^{\mathbf{j}\alpha}\delta(f-f_0)\,. \tag{17}$$

C. Discrete-time transforms

The discrete-time equivalent of the Fourier transform, the *Discrete-Time Fourier Transform* (*DTFT*), is defined by,

$$X(e^{\mathbf{j}\theta}) = \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}n\theta} \tag{18}$$

where $x[n]$ represents a discrete-time sequence, and $X(e^{\mathbf{j}\theta})$ is the DTFT. The *Inverse Discrete-Time Fourier Transform* (*IDTFT*) is defined by,

$$x[n] = \frac{1}{2\pi}\int_{\theta_0}^{(\theta_0+2\pi)} X(e^{\mathbf{j}\theta})e^{\mathbf{j}n\theta}d\theta \tag{19}$$

In both cases θ now represents the frequency variable (analogous to f in the continuous-time domain). Note that the DTFT is a continuous function of θ. In future lectures, we will see how to relate the frequency variable θ to the real frequency f, and how the DTFT is very closely related to the *frequency response* of discrete-time FIR systems.

For finite-length sequences, the *Discrete Fourier Transform* (*DFT*) is defined by,

$$X(k) = \sum_{n=0}^{N-1} x[n]e^{-\mathbf{j}2\pi nk/N} \tag{20}$$

where N is the length of sequence $x[n]$. Note that the DFT is also a discrete-(frequency) sequence of length N. Comparing the DTFT and the DFT, we observe that the DFT is a sampled version of the DTFT:

$$X(k) = X(e^{j\theta})\big|_{\theta = 2\pi k/N} \tag{21}$$

The *Inverse Discrete Fourier Transform* (*IDFT*) is defined by,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{\mathbf{j}2\pi nk/N} \tag{22}$$

and allows us to completely recover the original discrete-time sequence $x[n]$ from the DFT $X(k)$. Finally, the *Fast Fourier Transform* (*FFT*) and *Inverse Fast Fourier Transform* (*IFFT*) are computationally efficient algorithms for computing the DFT and IDFT, respectively.

D. Conclusion

We will have much more to say about the discrete-time transforms in the previous section, and how to interpret them — that is, how to relate the frequency variable θ and frequency index k to corresponding real frequencies for discrete-time signals sampled at some sampling frequency f_s. First though, for completeness sake, we will briefly examine the Fourier series and continuous-time Fourier transform, before transitioning to the discrete-time transforms.