Fourier Series to Fourier Transform

1. Introduction

In these notes, we continue our discussion of the Fourier series and relate it to the continuous-time Fourier transform through a specific example. We then conclude by looking at the frequency representation (Fourier transform) of several time-limited signals of different duration to observe an important property of the Fourier transform and the frequency spectrum of continuous-time signals.

2. Complex exponential to sinusoidal representation

A. Introduction

Previously, we have computed the Fourier coefficients X_k for some periodic waveforms, and then explicitly derived a real representation (in terms of cosines and sines) for $x(t)$ from the complex exponential form of the Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathbf{j}2\pi k f_0 t} \tag{1}$$

For example, in the lecture #13 notes, we derived the following Fourier coefficients for a triangle wave (symmetric about the vertical axis),

$$X_k = \begin{cases} 2/(\pi k)^2 & k = odd \\ 0 & k = even, k \neq 0 \\ 1/2 & k = 0 \end{cases} \tag{2}$$

and converted the complex exponential series,

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathbf{j}2\pi k t} \tag{3}$$

to the following sinusoidal representation:

$$x(t) = 1/2 + \sum_k \frac{4}{(\pi k)^2}\cos(2\pi k t),\ k \in \{1, 3, 5, \ldots\}. \tag{4}$$

Here, we will show that once we have computed the Fourier coefficients, we can directly represent real-valued periodic signals $x(t)$ in sinusoidal form using the following relationship:

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(2\pi k f_0 t + \angle X_k) \tag{5}$$

B. Proof

To show that equation (5) follows from equation (1), we will use the following property of the Fourier coefficients:

$$X_{-k} = X_k^* \tag{6}$$

That is, X_{-k} is the complex conjugate of X_k. We begin with equation (1) and use Euler's equation:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathbf{j}2\pi k f_0 t} \tag{7}$$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \cos(2\pi k f_0 t) + \mathbf{j} X_k \sin(2\pi k f_0 t) \tag{8}$$

$$x(t) = X_0 + \sum_{k=1}^{\infty} X_k \cos(2\pi k f_0 t) + X_{-k} \cos(2\pi(-k) f_0 t) + \mathbf{j}(X_k \sin(2\pi k f_0 t) + X_{-k} \sin(2\pi(-k) f_0 t)) \tag{9}$$

Now we use equation (6) and the even/odd property of the cosine/sine functions, respectively:

$$\cos(t) = \cos(-t) \tag{10}$$

$$\sin(t) = -\sin(-t) \tag{11}$$

Simplifying equation (9):

$$x(t) = X_0 + \sum_{k=1}^{\infty} X_k \cos(2\pi k f_0 t) + X_k^* \cos(2\pi k f_0 t) + \mathbf{j}(X_k \sin(2\pi k f_0 t) - X_k^* \sin(2\pi k f_0 t)) \tag{12}$$

$$x(t) = X_0 + \sum_{k=1}^{\infty} (X_k + X_k^*) \cos(2\pi k f_0 t) + \mathbf{j}(X_k - X_k^*) \sin(2\pi k f_0 t) \tag{13}$$

Note the following simplifications:

$$X_k + X_k^* = 2Re[X_k] \tag{14}$$

$$X_k - X_k^* = \mathbf{j} 2Im[X_k] \tag{15}$$

so that equation (13) simplifies to:

$$x(t) = X_0 + \sum_{k=1}^{\infty} 2Re[X_k] \cos(2\pi k f_0 t) - 2Im[X_k] \sin(2\pi k f_0 t) \tag{16}$$

$$x(t) = X_0 + 2 \sum_{k=1}^{\infty} Re[X_k] \cos(2\pi k f_0 t) - Im[X_k] \sin(2\pi k f_0 t) \tag{17}$$

We will now show that for any complex number z, the following relationship is true:

$$Re[z] \cos(t) - Im[z] \sin(t) = |z| \cos(t + \angle z) \tag{18}$$

Let $z = re^{\mathbf{j}\theta}$, so that:

$$Re[z] = r\cos(\theta) \tag{19}$$

$$Im[z] = r\sin(\theta) \tag{20}$$

$$|z| = r \tag{21}$$

$$\angle z = \theta . \tag{22}$$

Plugging (19) through (21) into (18):

$$r\cos(\theta)\cos(t) - r\sin(\theta)\sin(t) = r\cos(t + \theta) \tag{23}$$

$$\cos(t + \theta) = \cos(\theta)\cos(t) - \sin(\theta)\sin(t) \tag{24}$$

Equation (24) is a well known trigonometric identity that we showed to be true (using complex exponentials) in the lecture #12 notes. Having verified equation (18), we can now simplify (17) to the following form:

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(2\pi k f_0 t + \angle X_k) \quad (25)$$

C. Examples

In previous notes, we derived the following Fourier coefficients for an odd square wave with period $T_0 = 1$ ($f_0 = 1$):

$$X_k = \begin{cases} 1/(\mathbf{j}\pi k) & k = odd \\ 0 & k = even \end{cases} \quad (26)$$

for which we have that:

$$|X_k| = 1/(\pi k),\ k = \{1, 3, 5, \ldots\} \quad (27)$$

$$\angle X_k = -\pi/2,\ k = \{1, 3, 5, \ldots\}. \quad (28)$$

Combining (27) and (28) with (25) we get:

$$x(t) = 2\sum_{k=odd} \frac{1}{\pi k}\cos(2\pi k t - \pi/2) \quad (29)$$

$$x(t) = \sum_{k=odd} \frac{2}{\pi k}\sin(2\pi k t). \quad (30)$$

In the previous notes, we also derived the following Fourier coefficients for an odd sawtooth wave with period $T_0 = 1$ ($f_0 = 1$):

$$X_k = \begin{cases} (\mathbf{j}(-1)^k)/(2\pi k) & k \neq 0 \\ 0 & k = 0 \end{cases} \quad (31)$$

for which we have that:

$$|X_k| = 1/(2\pi k),\ k > 0 \quad (32)$$

$$\angle X_k = \begin{cases} -\pi/2 & k = odd \\ \pi/2 & k = even \end{cases}. \quad (33)$$

Combining (32) and (33) with (25) we get:

$$x(t) = 2\left[\sum_{k=odd} \frac{1}{(2\pi k)}\cos(2\pi k t - \pi/2) + \sum_{k=even} \frac{1}{(2\pi k)}\cos(2\pi k t + \pi/2)\right] \quad (34)$$

$$x(t) = \sum_{k=odd} \frac{1}{(\pi k)}\sin(2\pi k t) + \sum_{k=even} \frac{-1}{(\pi k)}\sin(2\pi k t) \quad (35)$$

$$x(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\pi k}\sin(2\pi k t). \quad (36)$$

Note that in both cases, the results agree with those arrived at explicitly in the lecture #13 notes.

3. Fourier series to Fourier transform

A. Introduction

Here we motivate the continuous Fourier transform as a limiting case of the Fourier series for $T_0 \to \infty$. We will do this by computing the Fourier series representation of a pulse train waveform $x(t)$ centered at $t = 0$ and varying the period of $x(t)$. In Figure 1, for example, we plot a pulse train waveform for various periods of increasing width. As $T_0 \to \infty$, the pulse train waveform approaches a single pulse centered at $t = 0$ with width equal to one.

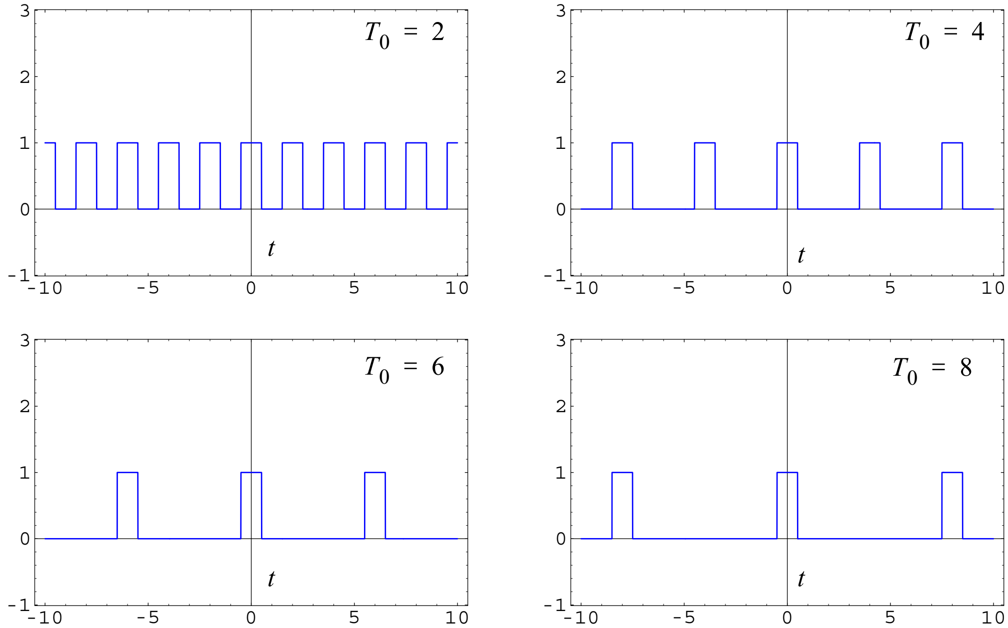

Figure 1

B. Fourier series representation

Here, we compute the Fourier series coefficients X_k for the pulse train wave $x(t)$ which is plotted in Figure 1 above for different periods T_0. Recall that the Fourier coefficient X_k is given by,

$$X_k = \frac{1}{T_0}\left[\int_{t_0}^{(t_0+T_0)} x(t)e^{-\mathbf{j}2\pi k f_0 t}dt\right] \tag{37}$$

For our problem, the integral in (37) reduces to:

$$X_k = \frac{1}{T_0}\int_{-1/2}^{1/2} e^{-\mathbf{j}2\pi k f_0 t}dt \tag{38}$$

First, we compute X_0:

$$\begin{aligned} X_0 &= \frac{1}{T_0}\int_{-1/2}^{1/2} dt \\ &= \left.\frac{t}{T_0}\right|_{t=-1/2}^{t=1/2} = \frac{1}{T_0} \end{aligned} \tag{39}$$

Next, we compute X_k, $k \neq 0$:

$$\begin{aligned}\frac{1}{T_0}\int_{-1/2}^{1/2} e^{-\mathbf{j}2\pi k f_0 t}dt &= \frac{1}{T_0}\left(\frac{1}{-\mathbf{j}2\pi f_0 k}\right)e^{-\mathbf{j}2\pi k f_0 t}\Big|_{t=-1/2}^{t=1/2} \\ &= \frac{1}{T_0}\left(\frac{1}{-\mathbf{j}2\pi f_0 k}\right)e^{-\mathbf{j}2\pi k f_0 (1/2)} - \frac{1}{T_0}\left(\frac{1}{-\mathbf{j}2\pi f_0 k}\right)e^{-\mathbf{j}2\pi k f_0(-1/2)} \\ &= \left(\frac{1}{-\mathbf{j}2\pi T_0 f_0 k}\right)e^{-\mathbf{j}\pi k f_0} - \left(\frac{1}{-\mathbf{j}2\pi T_0 f_0 k}\right)e^{\mathbf{j}\pi k f_0} \\ &= \left(\frac{1}{\pi k}\right)\left(\frac{e^{\mathbf{j}\pi k f_0} - e^{-\mathbf{j}\pi k f_0}}{\mathbf{j}2}\right) \\ &= \frac{\sin(\pi k f_0)}{\pi k}\end{aligned} \tag{40}$$

Thus, the Fourier coefficients X_k are given by,

$$X_k = \frac{\sin(\pi k f_0)}{\pi k} = \frac{\sin(\pi k / T_0)}{\pi k},\ k \neq 0, \tag{41}$$

$$X_0 = 1/T_0. \tag{42}$$

Since X_k is strictly real, we can now plot X_k as a function of frequency $f = kf_0 = k/T_0$ for different values of T_0. In Figure 2 below, we plot the Fourier coefficients as a function of frequency for the pulse train waveforms in Figure 1.

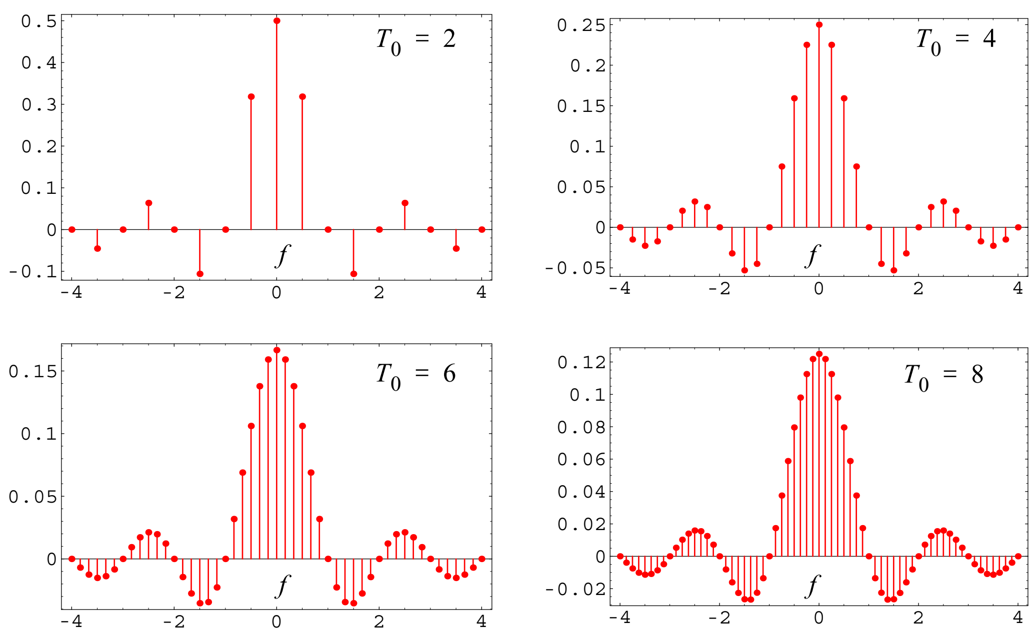

Figure 2

C. Fourier transform

We now compute the Fourier transform of a rectangular pulse $x_p(t)$ centered at $t = 0$, as plotted in Figure 3 below, which corresponds to a pulse train waveform with $T_0 \rightarrow \infty$.

Recall that the Fourier transform of a continuous-time signal $x(t)$ is given by,

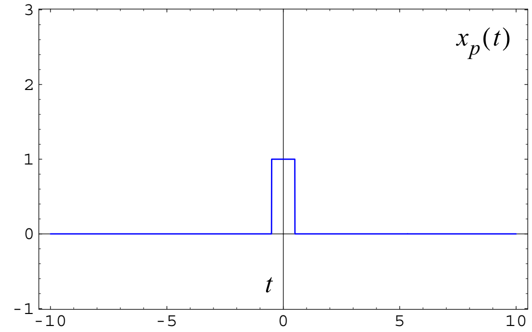

Figure 3

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-\mathbf{j}2\pi ft}dt \tag{43}$$

For the pulse in Figure 3, this integral is relatively easy to evaluate:

$$X(f) = \int_{-1/2}^{1/2} e^{-\mathbf{j}2\pi ft}dt \tag{44}$$

$$X(f) = \left.\frac{e^{-\mathbf{j}2\pi ft}}{-\mathbf{j}2\pi f}\right|_{t=-1/2}^{t=1/2} = \left(\frac{e^{-\mathbf{j}2\pi f(1/2)}}{-\mathbf{j}2\pi f}\right) - \left(\frac{e^{-\mathbf{j}2\pi f(-1/2)}}{-\mathbf{j}2\pi f}\right) \tag{45}$$

$$X(f) = \left(\frac{1}{-\mathbf{j}2\pi f}\right)(e^{-\mathbf{j}\pi f} - e^{\mathbf{j}\pi f}) \tag{46}$$

$$X(f) = \left(\frac{1}{\pi f}\right)\left(\frac{e^{\mathbf{j}\pi f} - e^{-\mathbf{j}\pi f}}{\mathbf{j}2}\right) = \frac{\sin(\pi f)}{\pi f} \tag{47}$$

This function is plotted in Figure 4 below. Note that the frequency spectrum of the single rectangular pulse closely resembles that of the pulse train waveform for large T_0. In fact, it is possible to compute the Fourier coefficients X_k of a periodic waveform from the Fourier transform $X(f)$ of a single period of that waveform, using the following relationship:

$$X_k = \frac{1}{T_0}X(f)\Big|_{f = kf_0} \tag{48}$$

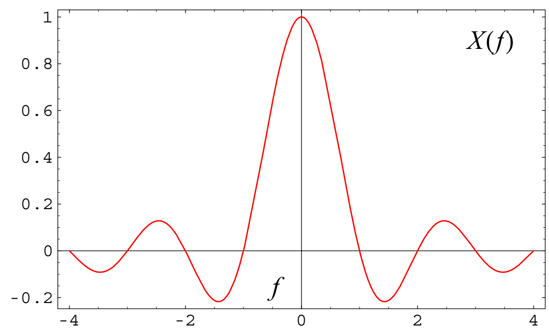

Figure 4

4. An interesting property of the Fourier transform

In this course, we will not delve deeply into the continuous-time Fourier transform and its properties (for much more details on the continuous Fourier transform see, for example, *Contemporary Linear Systems: Chapter 5* by R. D. Strum and D. E. Kirk). However, there is one interesting property of the time/frequency representation of signals that is worth exploring further here.

In Figure 5 we plot the Fourier transform $X(f)$ (frequency spectrum) of rectangular pulses $x(t)$ with varying widths. Note that the more spread out the signal is in the time domain, the more compressed it appears in the frequency domain, and vis versa. As a general rule, continuous-time signals that have *finite duration* in time will have nonzero frequency content throughout the frequency range from positive to negative infinity. Conversely, continuous-time signals that are *band-limited* in the frequency domain, will have infinite duration throughout time from positive to negative infinity.

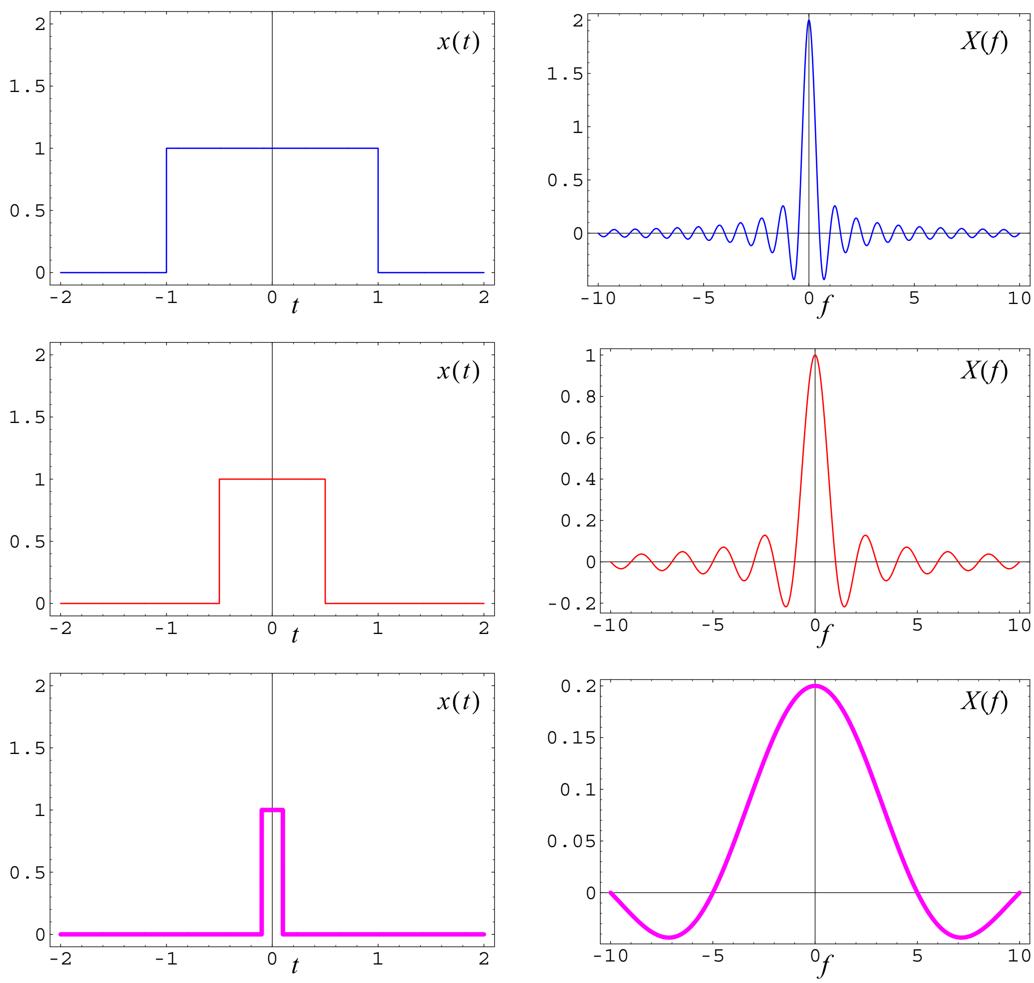

Figure 5

5. Conclusion

The *Mathematica* notebook "ctft.nb" was used to generate the examples in this set of notes. Next time, we will wrap up our discussion of the continuous-time Fourier transform and then transition to our exploration of the discrete-time Fourier transforms.