Fourier Series Examples

1. Introduction

In these notes, we derive in detail the Fourier series representation of several continuous-time periodic waveforms. Recall that we can write almost any periodic, continuous-time signal $x(t)$ as an infinite sum of *harmonically* related complex exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathbf{j}2\pi k f_0 t} \tag{1}$$

where,

$$X_k = \frac{1}{T_0}\left[\int_{t_0}^{(t_0+T_0)} x(t) e^{-\mathbf{j}2\pi k f_0 t} dt\right] = k\text{th Fourier coefficient,} \tag{2}$$

$$T_0 = \text{period of } x(t) \text{ (fundamental period), and,} \tag{3}$$

$$f_0 = 1/T_0 = \text{fundamental frequency of } x(t)\,. \tag{4}$$

For three different examples (triangle wave, sawtooth wave and square wave), we will compute the Fourier coefficients X_k as defined by equation (2), plot the resulting truncated Fourier series,

$$x'(t) = \sum_{k=-n}^{n} X_k e^{\mathbf{j}2\pi k f_0 t} \tag{5}$$

and the frequency-domain representation of each time-domain signal.

2. Example #1: triangle wave

Here, we compute the Fourier series coefficients X_k for the triangle wave $x(t)$ plotted in Figure 1 below. The functional representation of one period $x_p(t)$ of the triangle wave $t \in [-1/2, 1/2]$ is given by,

$$x_p(t) = \begin{cases} 2t+1 & t \in [-1/2, 0] \\ -2t+1 & t \in [0, 1/2] \end{cases} \tag{6}$$

The fundamental period T_0 and frequency f_0 are given by,

$$T_0 = 1\,,\; f_0 = 1/T_0 = 1 \tag{7}$$

Therefore, equation (2) for this problem is given by,

$$X_k = \int_{-1/2}^{1/2} x(t) e^{-\mathbf{j}2\pi k t} dt \tag{8}$$

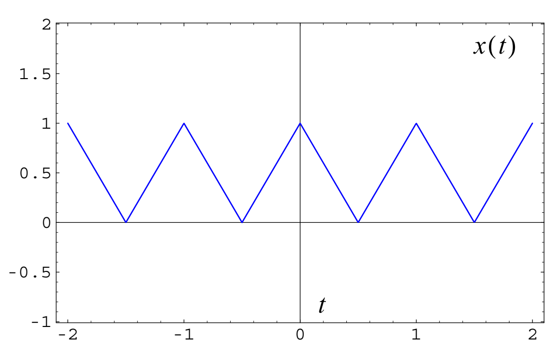

Figure 1

$$X_k = \int_{-1/2}^{0}(2t+1)e^{-\mathbf{j}2\pi kt}dt + \int_{0}^{1/2}(-2t+1)e^{-\mathbf{j}2\pi kt}dt \tag{9}$$

First, we compute X_0:

$$\begin{aligned} X_0 &= \int_{-1/2}^{0}(2t+1)dt + \int_{0}^{1/2}(-2t+1)dt \\ &= (t^2+t)\Big|_{t=-1/2}^{t=0} + (-t^2+t)\Big|_{t=0}^{t=1/2} \\ &= -(1/4-1/2)+(-1/4+1/2) \\ &= 1/2 \end{aligned} \tag{10}$$

Note that X_0 is simply the average of the function $x(t)$ for one period. Next, we compute X_k, $k \neq 0$. Using the following integral identity,

$$\int te^{at}dt = \frac{te^{at}}{a} - \frac{e^{at}}{a^2} \tag{11}$$

we will compute each term in equation (9) separately and then combine the results:

$$\begin{aligned} \int_{-1/2}^{0}(2t+1)e^{-\mathbf{j}2\pi kt}dt &= \left(\frac{2te^{-\mathbf{j}2\pi kt}}{-\mathbf{j}2\pi k} - \frac{2e^{-\mathbf{j}2\pi kt}}{(-\mathbf{j}2\pi k)^2}\right) + \left(\frac{e^{-\mathbf{j}2\pi kt}}{-\mathbf{j}2\pi k}\right)\Bigg|_{t=-1/2}^{t=0} \\ &= \left[-\frac{2}{(-\mathbf{j}2\pi k)^2} + \frac{1}{-\mathbf{j}2\pi k}\right] - \\ &\quad \left[\frac{2(-1/2)e^{-\mathbf{j}2\pi k(-1/2)}}{-\mathbf{j}2\pi k} - \frac{2e^{-\mathbf{j}2\pi k(-1/2)}}{(-\mathbf{j}2\pi k)^2} + \frac{e^{-\mathbf{j}2\pi k(-1/2)}}{-\mathbf{j}2\pi k}\right] \\ &= \left[-\frac{2}{(-\mathbf{j}2\pi k)^2} + \frac{1}{-\mathbf{j}2\pi k}\right] - \left[\frac{-e^{\mathbf{j}\pi k}}{-\mathbf{j}2\pi k} - \frac{2e^{\mathbf{j}\pi k}}{(-\mathbf{j}2\pi k)^2} + \frac{e^{\mathbf{j}\pi k}}{-\mathbf{j}2\pi k}\right] \\ &= -\frac{2}{(-\mathbf{j}2\pi k)^2} + \frac{1}{-\mathbf{j}2\pi k} + \frac{2e^{\mathbf{j}\pi k}}{(-\mathbf{j}2\pi k)^2} \\ &= \frac{1}{2(\pi k)^2} + \frac{\mathbf{j}}{2\pi k} - \frac{e^{\mathbf{j}\pi k}}{2(\pi k)^2} \end{aligned} \tag{12}$$

$$\begin{aligned} \int_{0}^{1/2}(-2t+1)e^{-\mathbf{j}2\pi kt}dt &= \left(\frac{-2te^{-\mathbf{j}2\pi kt}}{-\mathbf{j}2\pi k} - \frac{-2e^{-\mathbf{j}2\pi kt}}{(-\mathbf{j}2\pi k)^2}\right) + \left(\frac{e^{-\mathbf{j}2\pi kt}}{-\mathbf{j}2\pi k}\right)\Bigg|_{t=0}^{t=1/2} \\ &= \left[\frac{-2(1/2)e^{-\mathbf{j}2\pi k(1/2)}}{-\mathbf{j}2\pi k} - \frac{-2e^{-\mathbf{j}2\pi k(1/2)}}{(-\mathbf{j}2\pi k)^2} + \frac{e^{-\mathbf{j}2\pi k(1/2)}}{-\mathbf{j}2\pi k}\right] - \\ &\quad \left[-\frac{-2}{(-\mathbf{j}2\pi k)^2} + \frac{1}{-\mathbf{j}2\pi k}\right] \\ &= \left[\frac{e^{-\mathbf{j}\pi k}}{\mathbf{j}2\pi k} + \frac{2e^{-\mathbf{j}\pi k}}{(-\mathbf{j}2\pi k)^2} - \frac{e^{-\mathbf{j}\pi k}}{\mathbf{j}2\pi k}\right] - \left[-\frac{-2}{(-\mathbf{j}2\pi k)^2} + \frac{1}{-\mathbf{j}2\pi k}\right] \\ &= -\frac{e^{-\mathbf{j}\pi k}}{2(\pi k)^2} + \frac{1}{2(\pi k)^2} - \frac{\mathbf{j}}{2\pi k} \end{aligned} \tag{13}$$

Combining the results of equations (12) and (13),

$$
\begin{aligned}
X_k &= \left[\frac{1}{2(\pi k)^2} + \frac{\mathbf{j}}{2\pi k} - \frac{e^{\mathbf{j}\pi k}}{2(\pi k)^2}\right] + \left[-\frac{e^{-\mathbf{j}\pi k}}{2(\pi k)^2} + \frac{1}{2(\pi k)^2} - \frac{\mathbf{j}}{2\pi k}\right] \\
&= \frac{1}{(\pi k)^2} - \left[\frac{e^{\mathbf{j}\pi k}}{2(\pi k)^2} + \frac{e^{-\mathbf{j}\pi k}}{2(\pi k)^2}\right] \\
&= \frac{1}{(\pi k)^2} - \left[\frac{1}{(\pi k)^2}\left(\frac{e^{\mathbf{j}\pi k} + e^{-\mathbf{j}\pi k}}{2}\right)\right] \\
&= \frac{1}{(\pi k)^2} - \frac{1}{(\pi k)^2}\cos(\pi k) \\
&= \frac{1}{(\pi k)^2}[1 - \cos(\pi k)]
\end{aligned}
\tag{14}
$$

Note that since,

$$
\cos(\pi k) = \begin{cases} -1 & k = odd \\ 1 & k = even \end{cases}
\tag{15}
$$

we can summarize the values of the Fourier coefficients as follows:

$$
X_k = \begin{cases} 2/(\pi k)^2 & k = odd \\ 0 & k = even, k \neq 0 \\ 1/2 & k = 0 \end{cases}
\tag{16}
$$

Now that we have computed the Fourier series coefficients, we can express $x(t)$ as the sum of sinusoids:

$$
x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathbf{j}2\pi kt}
\tag{17}
$$

$$
x(t) = 1/2 + \sum_k \left(\frac{2e^{\mathbf{j}2\pi kt}}{(\pi k)^2} + \frac{2e^{-\mathbf{j}2\pi kt}}{(\pi(-k))^2}\right), \; k \in \{1, 3, 5, \ldots\}
\tag{18}
$$

$$
x(t) = 1/2 + \sum_k \frac{2}{(\pi k)^2}[e^{\mathbf{j}2\pi kt} + e^{-\mathbf{j}2\pi kt}], \; k \in \{1, 3, 5, \ldots\}
\tag{19}
$$

$$
x(t) = 1/2 + \sum_k \frac{4}{(\pi k)^2}\left[\frac{e^{\mathbf{j}2\pi kt} + e^{-\mathbf{j}2\pi kt}}{2}\right], \; k \in \{1, 3, 5, \ldots\}
\tag{20}
$$

$$
x(t) = 1/2 + \sum_k \frac{4}{(\pi k)^2}\cos(2\pi kt), \; k \in \{1, 3, 5, \ldots\}
\tag{21}
$$

To see how well equation (21) approximates $x(t)$ we plot the truncated Fourier series for $k \in \{1, 3, 5\}$,

$$
x'(t) \approx \frac{1}{2} + \frac{4}{\pi^2}\cos(2\pi t) + \frac{4}{(3\pi)^2}\cos(6\pi t) + \frac{4}{(5\pi)^2}\cos(10\pi t)
\tag{22}
$$

in Figure 2 below.

Here we make a few comments about our result in equation (21). First note that the Fourier series consists only of a constant offset ($1/2$) and *cosine* terms. This should be expected, since both the triangle wave, as defined, and the cosine function are examples of *even* functions, which obey the following property:

$$
x(t) = x(-t)
\tag{23}
$$

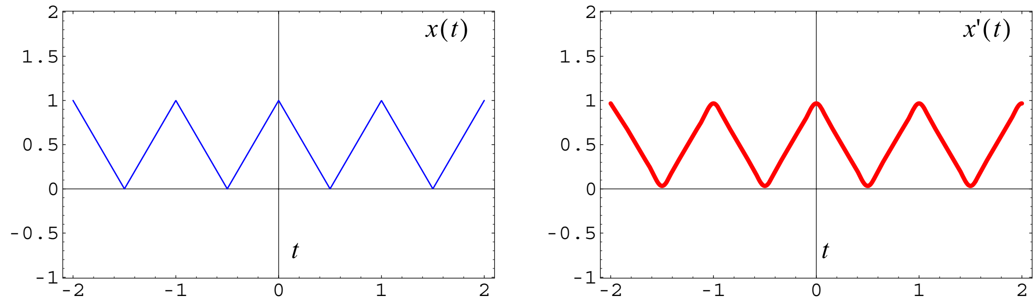

Figure 2

Second, we can view the Fourier series representation of $x(t)$ in the frequency domain by plotting $|X_k|$ and $\arg(X_k)$ as a function of $f = kf_0$. For this example, all the Fourier coefficients are strictly real (i.e. not complex), so that we can completely represent the frequency spectrum of the triangle wave by plotting X_k, as is done in Figure 3 below, instead of plotting $|X_k|$ and $\arg(X_k)$ separately.

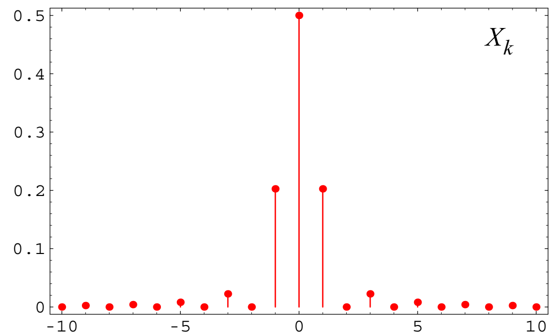

Figure 3

We can relate the frequency plot in Figure 3 to the Fourier transform $X(f)$ of the signal $x(t)$ using the Fourier transform pair,

$$A\cos(2\pi f_0 t) \Leftrightarrow \frac{A}{2}\delta(f+f_0) + \frac{A}{2}\delta(f-f_0) \tag{24}$$

which we have previously shown. Combining (24) with the Fourier series in (21), we get that:

$$X(f) = (1/2)\delta(f) + \sum_k \frac{2}{(\pi k)^2}[\delta(f+kf_0) + \delta(f-kf_0)],\ k \in \{1, 3, 5, \ldots\}. \tag{25}$$

3. Example #2: sawtooth wave

Here, we compute the Fourier series coefficients X_k for the sawtooth wave $x(t)$ plotted in Figure 4 below. The functional representation of one period $x_p(t)$ of the sawtooth wave $t \in [-1/2, 1/2]$ is given by,

$$x_p(t) = t,\ t \in [-1/2, 1/2] \tag{26}$$

The fundamental period T_0 and frequency f_0 are given by,

$$T_0 = 1,\ f_0 = 1/T_0 = 1 \tag{27}$$

Therefore, equation (2) for this problem is given by,

$$X_k = \int_{-1/2}^{1/2} x(t) e^{-\mathbf{j}2\pi kt} dt \tag{28}$$

$$X_k = \int_{-1/2}^{1/2} t e^{-\mathbf{j}2\pi kt} dt \tag{29}$$

First, we compute X_0:

$$\begin{aligned} X_0 &= \int_{-1/2}^{1/2} t dt \\ &= \left.\frac{t^2}{2}\right|_{t=-1/2}^{t=1/2} \\ &= 1/4 - 1/4 = 0 \end{aligned} \tag{30}$$

Note that X_0 is simply the average of the function $x(t)$ for one period. Next, we compute X_k, $k \neq 0$, where we will again make use of the following integral identity:

$$\int t e^{at} dt = \frac{te^{at}}{a} - \frac{e^{at}}{a^2} \tag{31}$$

$$\begin{aligned} \int_{-1/2}^{1/2} t e^{-\mathbf{j}2\pi kt} dt &= \left.\left(\frac{te^{-\mathbf{j}2\pi kt}}{-\mathbf{j}2\pi k} - \frac{e^{-\mathbf{j}2\pi kt}}{(-\mathbf{j}2\pi k)^2}\right)\right|_{t=-1/2}^{t=1/2} \\ &= \left[\frac{(1/2)e^{-\mathbf{j}2\pi k(1/2)}}{-\mathbf{j}2\pi k} - \frac{e^{-\mathbf{j}2\pi k(1/2)}}{(-\mathbf{j}2\pi k)^2}\right] - \left[\frac{(-1/2)e^{-\mathbf{j}2\pi k(-1/2)}}{-\mathbf{j}2\pi k} - \frac{e^{-\mathbf{j}2\pi k(-1/2)}}{(-\mathbf{j}2\pi k)^2}\right] \\ &= \frac{(1/2)e^{-\mathbf{j}\pi k}}{-\mathbf{j}2\pi k} + \frac{(1/2)e^{\mathbf{j}\pi k}}{-\mathbf{j}2\pi k} - \frac{e^{-\mathbf{j}\pi k}}{(-\mathbf{j}2\pi k)^2} + \frac{e^{\mathbf{j}\pi k}}{(-\mathbf{j}2\pi k)^2} \\ &= \frac{1}{-\mathbf{j}2\pi k}\cos(\pi k) + \frac{\mathbf{j}2}{(-\mathbf{j}2\pi k)^2}\sin(\pi k) \\ &= \frac{\mathbf{j}}{2\pi k}\cos(\pi k) - \frac{\mathbf{j}}{2(\pi k)^2}\sin(\pi k) \end{aligned} \tag{32}$$

Note that since,

$$\cos(\pi k) = \begin{cases} -1 & k = odd \\ 1 & k = even \end{cases} \tag{33}$$

$$\sin(\pi k) = 0,\ k \in \{\ldots, -2, -1, 1, 2, \ldots\} \tag{34}$$

we can summarize the values of the Fourier coefficients as follows:

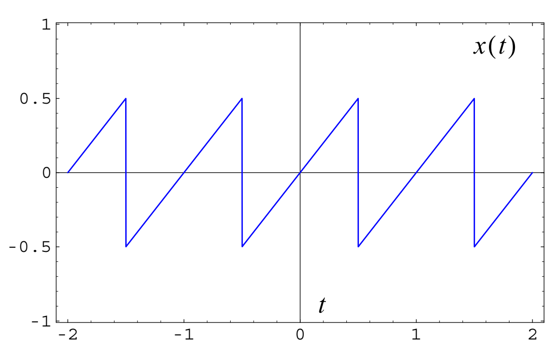

Figure 4

$$X_k = \frac{\mathbf{j}}{2\pi k}\cos(\pi k) - \frac{\mathbf{j}}{2(\pi k)^2}\sin(\pi k) \quad (35)$$

$$X_k = \begin{cases} (\mathbf{j}(-1)^k)/(2\pi k) & k \neq 0 \\ 0 & k = 0 \end{cases} \quad (36)$$

Now that we have computed the Fourier series coefficients, we can express $x(t)$ as the sum of sinusoids:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathbf{j}2\pi kt} \quad (37)$$

$$x(t) = \sum_{k=1}^{\infty} \left(\frac{\mathbf{j}(-1)^k}{2\pi k} e^{\mathbf{j}2\pi kt} + \frac{\mathbf{j}(-1)^{-k}}{2\pi(-k)} e^{-\mathbf{j}2\pi kt} \right) \quad (38)$$

$$x(t) = \sum_{k=1}^{\infty} \frac{\mathbf{j}(-1)^k}{2\pi k} (e^{\mathbf{j}2\pi kt} - e^{-\mathbf{j}2\pi kt}) \quad (39)$$

$$x(t) = \sum_{k=1}^{\infty} \frac{(\mathbf{j}2)\mathbf{j}(-1)^k}{2\pi k} \left[\frac{e^{\mathbf{j}2\pi kt} - e^{-\mathbf{j}2\pi kt}}{\mathbf{j}2} \right] \quad (40)$$

$$x(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\pi k} \sin(2\pi kt) \quad (41)$$

To see how well equation (41) approximates $x(t)$ we plot the truncated Fourier series for $k \in \{1, 2, 3, 4, 5\}$,

$$x'(t) \approx \frac{1}{\pi}\sin(2\pi t) - \frac{1}{2\pi}\sin(4\pi t) + \frac{1}{3\pi}\sin(6\pi t) - \frac{1}{4\pi}\sin(8\pi t) + \frac{1}{5\pi}\sin(10\pi t) \quad (42)$$

in Figure 5 below.

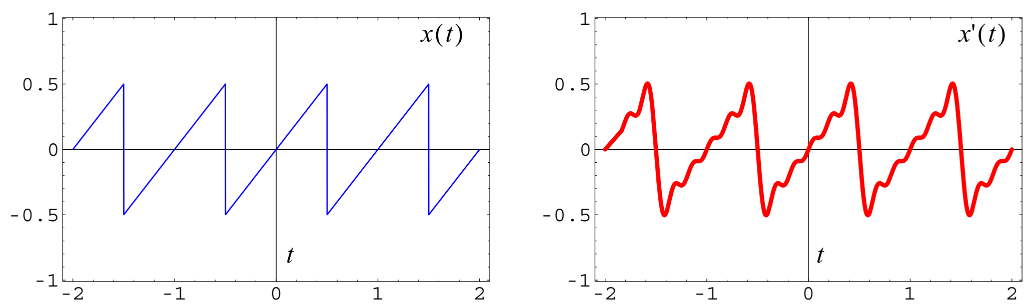

Figure 5

Here we make a few comments about our result in equation (41). First note that this Fourier series consists only of *sine* terms. This should be expected, since both the sawtooth wave, as defined, and the sine function are examples of *odd* functions, which obey the following property:

$$x(t) = -x(-t) \quad (43)$$

Second, the approximation in (42) does not seem nearly as accurate as was the approximation for the triangle wave in the previous section. This is so, because unlike the continuous triangle wave, the sawtooth wave has discontinuities at discrete intervals. It should not be surprising that it is significantly more difficult to model a discontinuous periodic signal with a sum of smooth, continuous sine waves, than it is to model a continuous

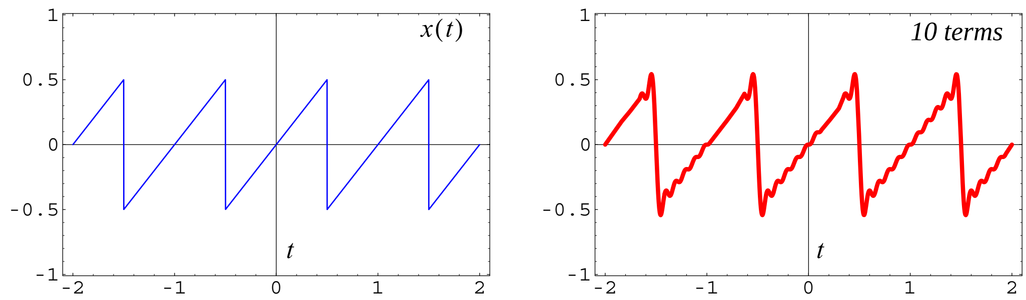

Figure 6

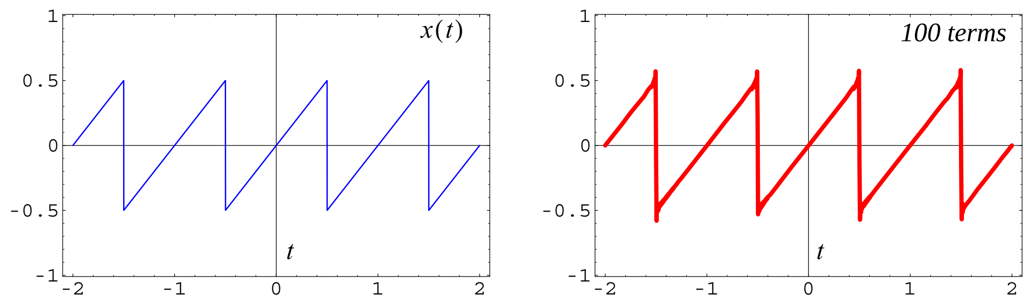

Figure 7

periodic signal. To see what the truncated Fourier series approximation looks like with more terms, we plot the truncated Fourier series with the first 10 and 100 terms in Figures 6 and 7, respectively.

Note that while the Fourier series approximation does seem to improve almost everywhere as we add more terms, there remains a stubborn (approximately 9%) visible error at the discontinuities of the sawtooth wave. This overshoot is known as *Gibbs phenomenon*, and will occur for any truncated Fourier series representation of discontinuous, periodic waveforms. We will see this again for the square wave in the next section.

Finally, we can view the Fourier series representation of $x(t)$ in the frequency domain by plotting $|X_k|$ and $\arg(X_k)$ as a function of $f = kf_0$. These two plots are shown in Figure 8 below. Note how the phase alternates between $\pi/2$ and $-\pi/2$, which is expected given the $(-1)^{k+1}$ term in the Fourier coefficients X_k.

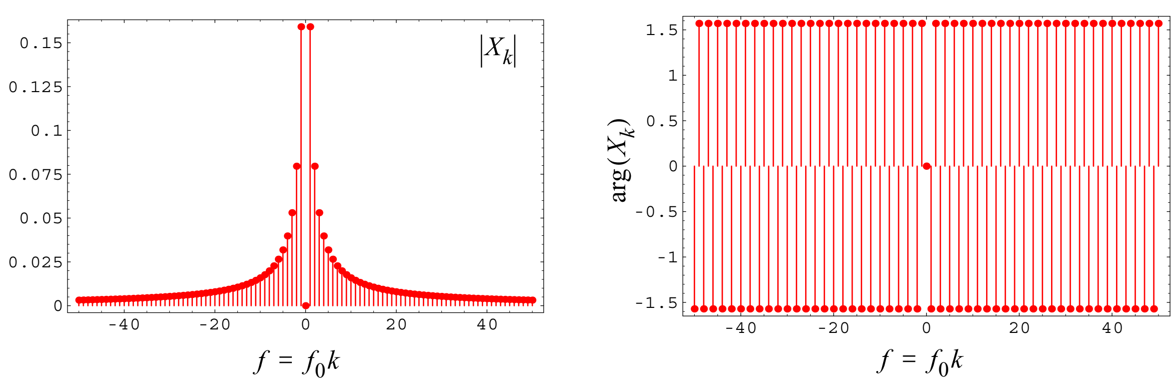

Figure 8

4. Example #3: square wave

Here, we compute the Fourier series coefficients X_k for the square wave $x(t)$ plotted in Figure 9 below. The functional representation of one period $x_p(t)$ of the square wave $t \in [-1/2, 1/2]$ is given by,

$$x_p(t) = \begin{cases} -1/2 & t \in [-1/2, 0] \\ 1/2 & t \in [0, 1/2] \end{cases} \quad (44)$$

The fundamental period T_0 and frequency f_0 are given by,

$$T_0 = 1 \, , f_0 = 1/T_0 = 1 \quad (45)$$

Therefore, equation (2) for this problem is given by,

$$X_k = \int_{-1/2}^{1/2} x(t) e^{-\mathbf{j}2\pi kt} dt \quad (46)$$

$$X_k = \int_{-1/2}^{0} (-1/2) e^{-\mathbf{j}2\pi kt} dt + \int_{0}^{1/2} (1/2) e^{-\mathbf{j}2\pi kt} dt \quad (47)$$

First, we compute X_0:

$$\begin{aligned} X_0 &= \int_{-1/2}^{0} (-1/2) dt + \int_{0}^{1/2} (1/2) dt \\ &= (-1/2t)\Big|_{t=-1/2}^{t=0} + (1/2t)\Big|_{t=0}^{t=1/2} \\ &= -(1/4) + (1/4) \\ &= 0 \end{aligned} \quad (48)$$

Note that X_0 is simply the average of the function $x(t)$ for one period. Next, we compute X_k, $k \neq 0$; we will compute each term in equation (48) separately and then combine the results.

$$\begin{aligned} \int_{-1/2}^{0} (-1/2) e^{-\mathbf{j}2\pi kt} dt &= \frac{(-1/2)}{-\mathbf{j}2\pi k} e^{-\mathbf{j}2\pi kt} \Big|_{t=-1/2}^{t=0} \\ &= \frac{(-1/2)}{-\mathbf{j}2\pi k} - \frac{(-1/2)}{-\mathbf{j}2\pi k} e^{\mathbf{j}\pi k} \\ &= \frac{1}{\mathbf{j}4\pi k}(1 - e^{\mathbf{j}\pi k}) \end{aligned} \quad (49)$$

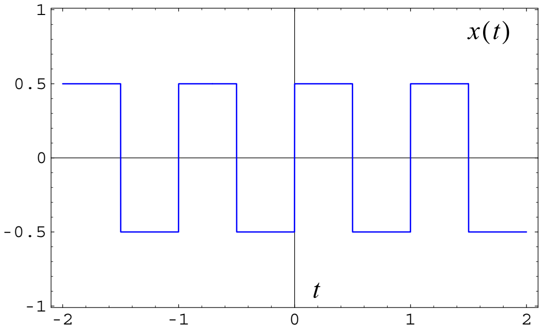

Figure 9

$$\begin{aligned}\int_0^{1/2}(1/2)e^{-\mathbf{j}2\pi kt}dt &= \frac{(1/2)}{-\mathbf{j}2\pi k}e^{-\mathbf{j}2\pi kt}\Big|_{t=0}^{t=1/2} \\ &= \frac{(1/2)}{-\mathbf{j}2\pi k}e^{-\mathbf{j}\pi k} - \frac{(1/2)}{-\mathbf{j}2\pi k} \\ &= \frac{1}{\mathbf{j}4\pi k}(1-e^{-\mathbf{j}\pi k})\end{aligned} \tag{50}$$

Combining the results of equations (49) and (50),

$$\begin{aligned}X_k &= \left[\frac{1}{\mathbf{j}4\pi k}(1-e^{\mathbf{j}\pi k})\right] + \left[\frac{1}{\mathbf{j}4\pi k}(1-e^{-\mathbf{j}\pi k})\right] \\ &= \frac{1}{\mathbf{j}2\pi k} - \left[\frac{e^{\mathbf{j}\pi k}+e^{-\mathbf{j}\pi k}}{\mathbf{j}4\pi k}\right] \\ &= \frac{1}{\mathbf{j}2\pi k} - \frac{1}{\mathbf{j}2\pi k}\left[\frac{e^{\mathbf{j}\pi k}+e^{-\mathbf{j}\pi k}}{2}\right] \\ &= \frac{1}{\mathbf{j}2\pi k} - \frac{1}{\mathbf{j}2\pi k}\cos(\pi k) \\ &= \frac{1}{\mathbf{j}2\pi k}[1-\cos(\pi k)]\end{aligned} \tag{51}$$

Note that since,

$$\cos(\pi k) = \begin{cases} -1 & k = odd \\ 1 & k = even \end{cases} \tag{52}$$

we can summarize the values of the Fourier coefficients as follows:

$$X_k = \begin{cases} 1/(\mathbf{j}\pi k) & k = odd \\ 0 & k = even \end{cases} \tag{53}$$

Now that we have computed the Fourier series coefficients, we can express $x(t)$ as the sum of sinusoids:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathbf{j}2\pi kt} \tag{54}$$

$$x(t) = \sum_k \left(\frac{e^{\mathbf{j}2\pi kt}}{\mathbf{j}\pi k} + \frac{e^{-\mathbf{j}2\pi kt}}{\mathbf{j}\pi(-k)}\right),\ k \in \{1, 3, 5, \ldots\} \tag{55}$$

$$x(t) = \sum_k \left(\frac{e^{\mathbf{j}2\pi kt}}{\mathbf{j}\pi k} - \frac{e^{-\mathbf{j}2\pi kt}}{\mathbf{j}\pi k}\right),\ k \in \{1, 3, 5, \ldots\} \tag{56}$$

$$x(t) = \sum_k \frac{2}{\pi k}\left(\frac{e^{\mathbf{j}2\pi kt} - e^{-\mathbf{j}2\pi kt}}{\mathbf{j}2}\right),\ k \in \{1, 3, 5, \ldots\} \tag{57}$$

$$x(t) = \sum_k \frac{2}{\pi k}\sin(2\pi kt),\ k \in \{1, 3, 5, \ldots\} \tag{58}$$

To see how well equation (50) approximates $x(t)$ we plot the truncated Fourier series for $k \in \{1, 3, 5\}$,

$$x'(t) \approx \frac{2}{\pi}\sin(2\pi t) + \frac{2}{3\pi}\sin(6\pi t) + \frac{2}{5\pi}\sin(10\pi t) \tag{59}$$

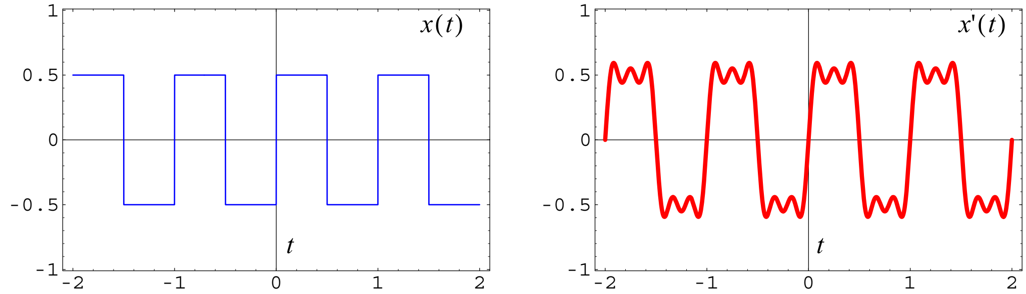

Figure 10

in Figure 10 above.

Here we make a few comments about our result in equation (58). As with the sawtooth wave, this Fourier series consists only of *sine* terms. Again, this should be expected, since both the square wave and the sine function are both examples of *odd* functions as defined in equation (43). Second, the approximation in (59) does not seem very accurate. To see what the truncated Fourier series approximation looks like with more terms, we plot the truncated Fourier series with the first 10 and 100 terms in Figures 11 and 12, respectively. As with the sawtooth wave, notice the overshoot at the discontinuities of the square wave (Gibbs phenomenon).

Finally, as we did for the previous two examples, we can view the Fourier series representation of $x(t)$ in the frequency domain by plotting $|X_k|$ and $\arg(X_k)$ as a function of $f = kf_0$. These two plots are shown in Figure 13 below. Note that for $k < 0$, the phase for nonzero X_k is $-\pi/2$, while for $k > 0$, the phase for nonzero X_k is $\pi/2$. This is easily seen if we rewrite (53) as,

$$X_k = 1/(\mathbf{j}\pi k) = -\mathbf{j}/(\pi k),\ k = odd. \quad (60)$$

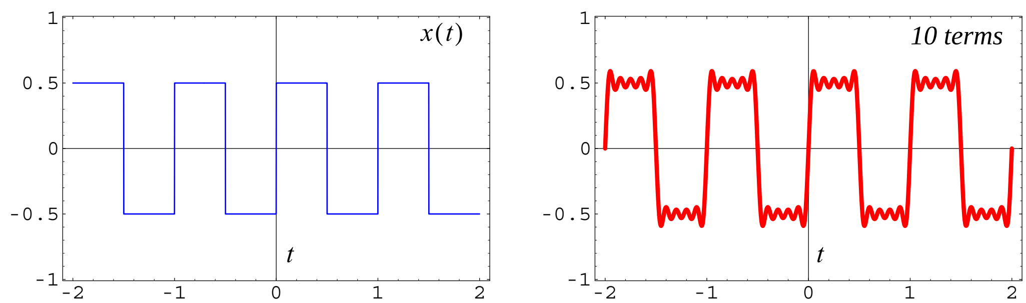

Figure 11

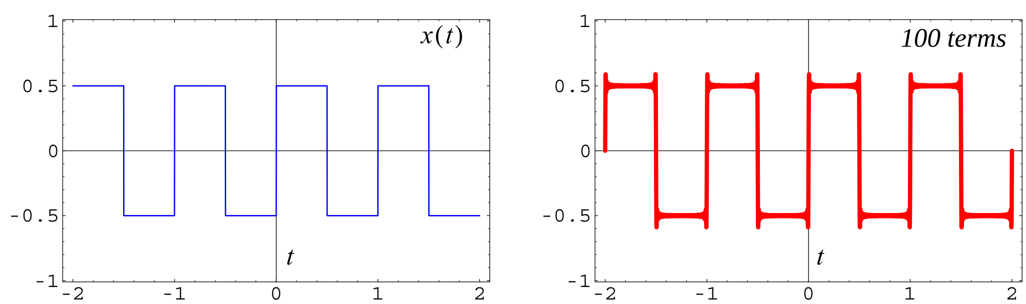

Figure 12

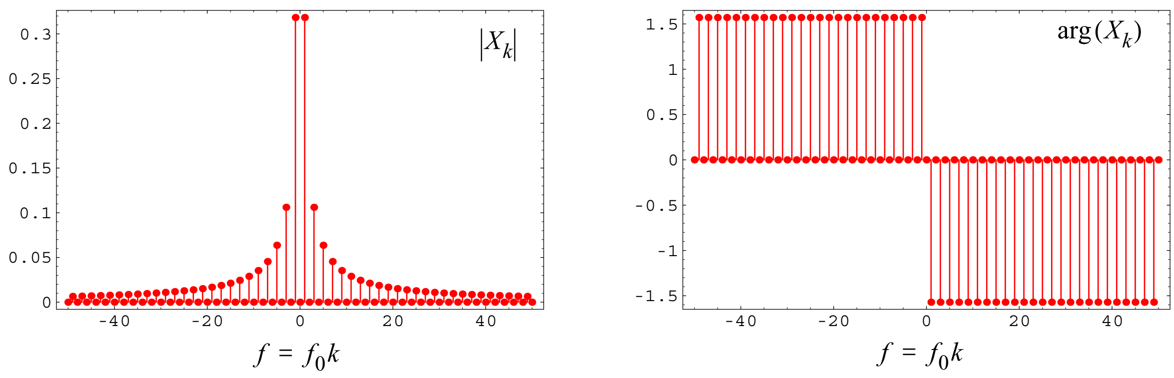

Figure 13

5. Conclusion

See the *Mathematica* notebook “fourier_series.nb” for all the examples in this set of notes. Next, time we will finish up with the Fourier series representation and show how the Fourier transform can be viewed as a limiting case of the Fourier series with $T_0 \to \infty$.