The Discrete Fourier Transform (DFT)

1. Introduction

In these notes, we introduce the *Discrete Fourier Transform (DFT)*, and show its relationship to the *discrete-time Fourier transform* (*DTFT*).

2. Relating the Discrete Fourier Transform (DFT) to the DTFT

A. Introduction

We will now tie the DTFT to the *discrete Fourier transform* (*DFT*), which is defined for finite-length sequences $x[n]$ of length N:

$$x[n] = \begin{cases} nonzero & 0 \le n \le N-1 \\ 0 & elsewhere \end{cases} \tag{1}$$

For such sequences, the DTFT is given by,

$$X(e^{\mathbf{j}\theta}) = \sum_{n=0}^{N-1} x[n]e^{-\mathbf{j}n\theta} \tag{2}$$

while the DFT is given by,

$$X(k) = \sum_{n=0}^{N-1} x[n]e^{-\mathbf{j}2\pi nk/N},\ 0 \le k \le N-1\ . \tag{3}$$

By comparing the two definitions in equations (1) and (2), we observe that the DFT is a sampled version of the DTFT, as given by the following relationship:

$$X(k) = X(e^{\mathbf{j}\theta})\big|_{\theta = 2\pi k/N} \tag{4}$$

Note that for a sequence $x[n]$ of length N, the DFT generates a list of N frequency coefficients $X(k)$. Given the frequency coefficients $X(k)$, it is possible to recover the original sequence $x[n]$ by applying the *inverse discrete Fourier transform* (*IDFT*) given by,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{\mathbf{j}2\pi nk/N},\ 0 \le n \le N-1\ . \tag{5}$$

B. Relating index k to frequencies f for sampled signals

Let us assume that a discrete-time sequence $x[n]$ of length N was generated by sampling a continuous-time signal $x_c(t)$ with sampling frequency f_s. One question we would like to answer is to what real frequencies f each index k corresponds. In our derivation below, we will use the previously given relationship between f and the frequency variable θ in the DTFT,

$$f = \frac{\theta f_s}{2\pi} \tag{6}$$

as well as the periodic property of the DTFT:

$$X(e^{\mathbf{j}(\theta + 2\pi)}) = X(e^{\mathbf{j}\theta})\ . \tag{7}$$

For now, we will also assume that N is even.We will first consider the case of $k = 0$, which corresponds to $\theta = 0$. Using relationship (6), this case corresponds to $f = 0\text{Hz}$. Second, let us consider the following range of k:

$$1 \le k \le (N/2 - 1)\,. \tag{8}$$

For these values of k, the corresponding range of θ is given by:

$$\frac{2\pi}{N} \le \theta \le \frac{2\pi(N/2 - 1)}{N} \tag{9}$$

$$\frac{2\pi}{N} \le \theta \le \pi - \frac{2\pi}{N} \tag{10}$$

Therefore this range of k corresponds to positive frequency values,

$$\left(\frac{2\pi}{N}\right)\left(\frac{f_s}{2\pi}\right) \le f \le \left(\pi - \frac{2\pi}{N}\right)\left(\frac{f_s}{2\pi}\right) \tag{11}$$

$$\frac{f_s}{N} \le f \le \left(\frac{f_s}{2} - \frac{f_s}{N}\right) \tag{12}$$

Third, let us consider the case of $k = N/2$, which corresponds to $\theta = \pi$, and, using the periodicity of $X(e^{\mathbf{j}\theta})$, $\theta = -\pi$. Using relationship (6), this value of k therefore corresponds to:

$$f = \pm f_s/2\,. \tag{13}$$

Finally, let us consider the following range of k:

$$N/2 + 1 \le k \le N - 1\,. \tag{14}$$

For these values of k, the corresponding range of θ is given by:

$$\frac{2\pi(N/2 + 1)}{N} \le \theta \le \frac{2\pi(N - 1)}{N} \tag{15}$$

$$\pi + \frac{2\pi}{N} \le \theta \le 2\pi - \frac{2\pi}{N} \tag{16}$$

Using the periodicity of $X(e^{\mathbf{j}\theta})$, we can subtract 2π from the values in (16) above, so that:

$$-\pi + \frac{2\pi}{N} \le \theta \le \frac{-2\pi}{N} \tag{17}$$

Therefore the range of k in equation (14) corresponds to negative frequency values,

$$\left(-\pi + \frac{2\pi}{N}\right)\left(\frac{f_s}{2\pi}\right) \le f \le \left(\frac{-2\pi}{N}\right)\left(\frac{f_s}{2\pi}\right) \tag{18}$$

$$\left(\frac{-f_s}{2} + \frac{f_s}{N}\right) \le f \le \frac{-f_s}{N} \tag{19}$$

The table below summarizes these results for N even, while Figure 1 illustrates the correspondence between k and f graphically for sample values $f_s = 10\text{Hz}$ and $N = 30$.

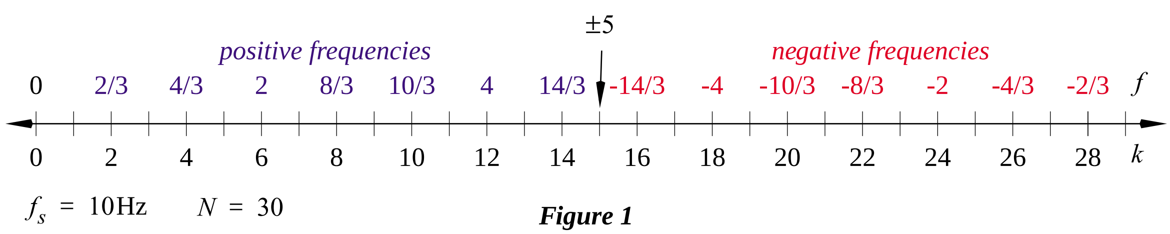


Figure 1

N = even		
index k	*frequency variable* θ	*corresponding frequency* f
$k = 0$	$\theta = 0$	$f = 0$
$1 \le k \le (N/2 - 1)$	$\frac{2\pi}{N} \le \theta \le \pi - \frac{2\pi}{N}$	$\frac{f_s}{N} \le f \le \left(\frac{f_s}{2} - \frac{f_s}{N}\right)$
$k = N/2$	$\theta = \pm\pi$	$f = \pm f_s/2$
$N/2 + 1 \le k \le N - 1$	$-\pi + \frac{2\pi}{N} \le \theta \le \frac{-2\pi}{N}$	$\left(\frac{-f_s}{2} + \frac{f_s}{N}\right) \le f \le \frac{-f_s}{N}$

A similar derivation yields the frequency correspondences when N is odd, as shown in the table below. Note that in both cases, the DFT gives us the frequency content of a discrete-time signal at discrete frequencies that are integer multiples of f_s/N in the frequency range $[-f_s/2, f_s/2]$.

N = odd		
index k	*frequency variable* θ	*corresponding frequency* f
$k = 0$	$\theta = 0$	$f = 0$
$1 \le k \le (N-1)/2$	$\frac{2\pi}{N} \le \theta \le \pi - \frac{\pi}{N}$	$\frac{f_s}{N} \le f \le \left(\frac{f_s}{2} - \frac{f_s}{2N}\right)$
$(N-1)/2 + 1 \le k \le N - 1$	$-\pi + \frac{\pi}{N} \le \theta \le \frac{-2\pi}{N}$	$\left(\frac{-f_s}{2} + \frac{f_s}{2N}\right) \le f \le \frac{-f_s}{N}$

3. DFT examples

Below, we illustrate some of the properties of the DFT through some simple examples. We specifically consider the six examples in the table below:

N	*continuous-time signal*	*sampling frequency*	*Figure*
$N = 30$ (3 seconds)	$x_1(t) = \cos(2\pi t)$	$f_s = 10\text{Hz}$	Figure 2
$N = 60$ (3 seconds)	$x_2(t) = \cos(2\pi t)$	$f_s = 20\text{Hz}$	Figure 3
$N = 30$ (3 seconds)	$x_3(t) = \cos[2\pi(7/6)t]$	$f_s = 10\text{Hz}$	Figure 4
$N = 60$ (3 seconds)	$x_4(t) = \cos[2\pi(7/6)t]$	$f_s = 20\text{Hz}$	Figure 5
$N = 30$ (3 seconds)	$x_5(t) = 1 + 2\cos(2\pi t) + 4\cos(4\pi t)$	$f_s = 10\text{Hz}$	Figure 6
$N = 30$ (3 seconds)	$x_6(t) = 1 + 2\cos[2\pi(7/6)t] + 4\cos[2\pi(13/6)t]$	$f_s = 10\text{Hz}$	Figure 7

In each case, the sampled sequence $x_m[n]$ is given by,

$$x_m[n] = \begin{cases} x_m(n/f_s) & 0 \le n < N \\ 0 & elsewhere \end{cases}, \; m \in \{1, 2, 3, 4, 5, 6\}. \tag{20}$$

For each example, we plot the DFT as a function of k ($|X(k)|$ and $\angle X(k)$) and as a function of frequency f, using the conversions in the previous tables; in the Figures, we denote this conversion from k to f by,

$$k \Rightarrow f \tag{21}$$

so that,

$$|X(k)|_{k \Rightarrow f} \text{ and } \angle X(k)|_{k \Rightarrow f} \tag{22}$$

denote the DFT as a function of real frequencies f. In the DFT graphs that are plotted as a function of f, the corresponding magnitude and phase DTFTs are superimposed using a dashed blue line. The table above specifies which Figure corresponds to which example.

From Figures 2 through 7, we make the following observations. First of all, the DFT is difficult to interpret when plotted as a function of k, since it is not readily apparent to which frequencies each of the $X(k)$ coefficients corresponds. Therefore, we rely on the conversion tables above to plot the frequency coefficients as a function of frequency f. Second, we observe that the DFT is indeed a sampled version of the DTFT.

Third, when a signal consists only of frequencies that are integer multiples of f_s/N, those frequencies have nonzero coefficients, while all other coefficients are zero (see Figures 2, 3 and 6). However, when a signal consists of frequencies that are *not* integer multiples of f_s/N, the DFT no longer samples the DTFT at its peaks and zero crossings, and therefore, the DFT coefficients are no longer zero for frequencies not in the original signal (see Figures 4, 5 and 7). In fact, the frequency content is spread out (i.e. nonzero) over the full range of DFT coefficients $X(k)$. As in the case of the DTFT, this spectral leakage is caused by *finite-length* sampling that occurs for any practical application. Increasing the sampling frequency, thereby generating longer discrete-time sequences for equivalent sampling times, reduces spectral leakage, but does not eliminate the problem.

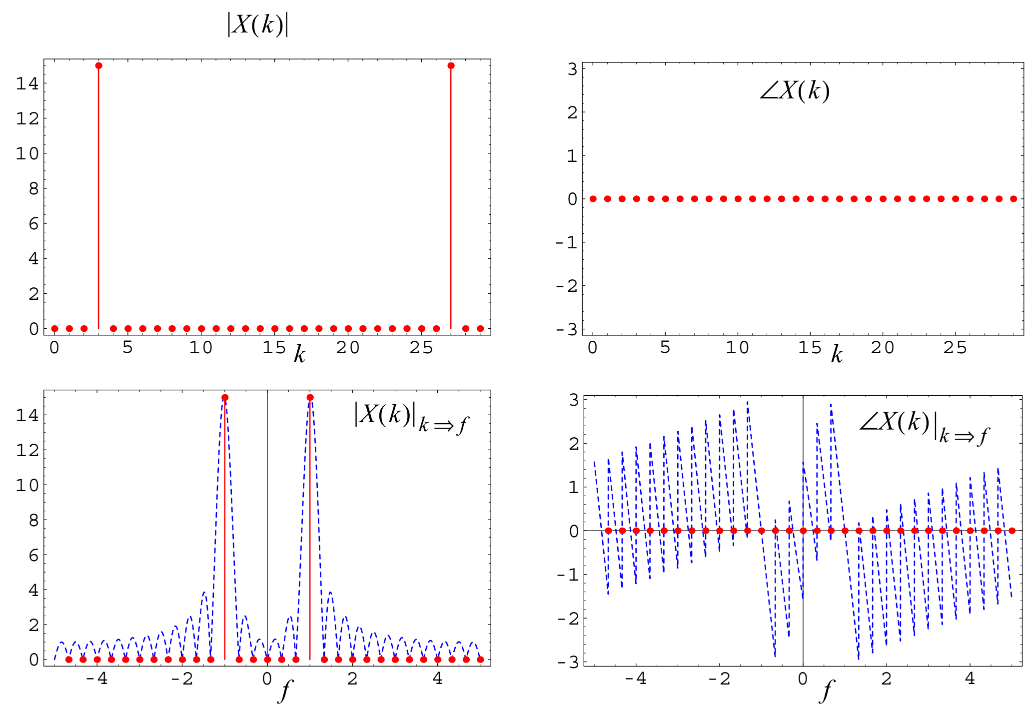
Finally, we note that for real-valued discrete-time signals $x[n]$, the magnitude of the DFT coefficients is an even function of f, while the phase of the DFT coefficients is an odd function of f. In terms of k, we can express this fact as follows:

$$|X(k)| = |X(N-k)|,\ k \neq 0, \text{ and,} \tag{23}$$

$$\angle X(k) = -\angle X(N-k),\ k \neq 0. \tag{24}$$

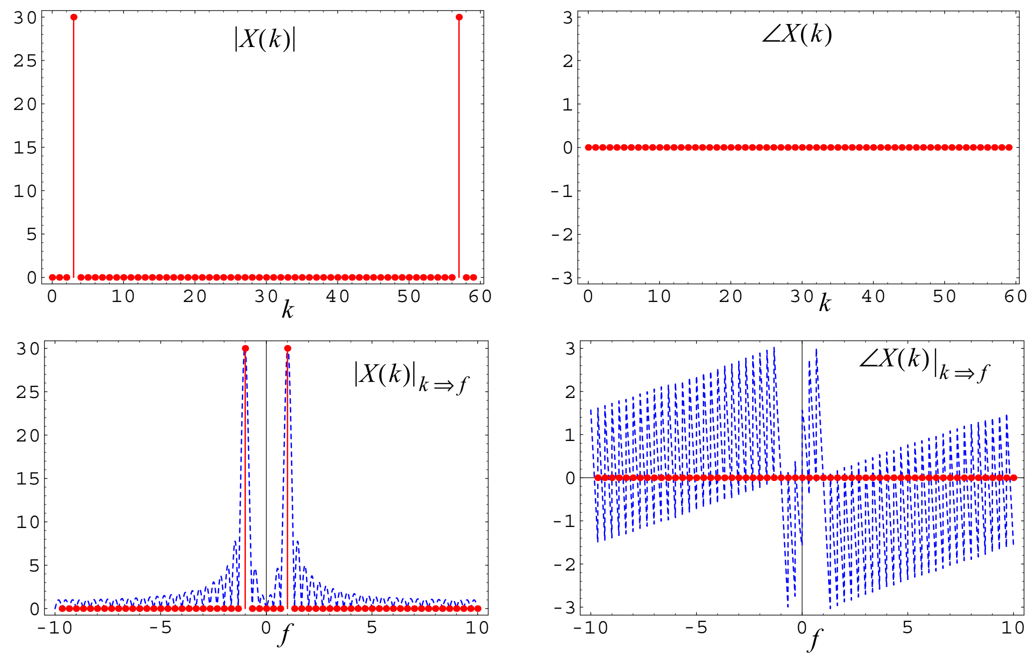
4. Conclusion

The *Mathematica* notebook “dft.nb” was used to generate the examples in this set of notes. In subsequent notes, we will continue our discussion of the DFT, including *windowing* methods that can alleviate the problem of spectral leakage. We will also talk about the FFT as a computationally efficient algorithm for computing the DFT, and revisit applications first introduced at the beginning of this course, in the context of our new understanding of frequency transforms.



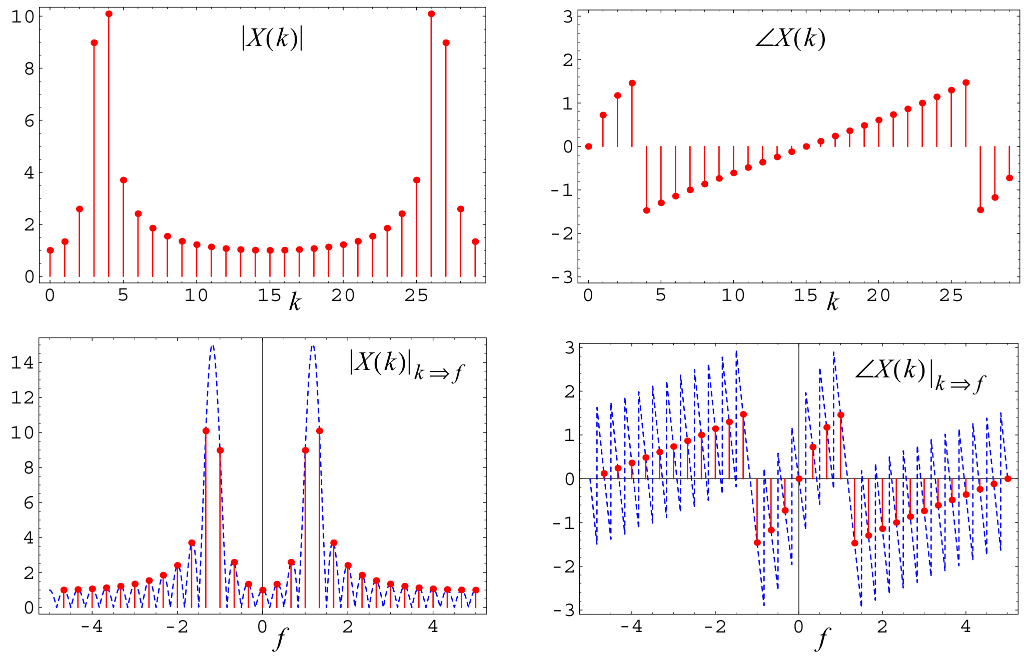
Dashed blue line = DTFT

Figure 2: DFT corresponding to $x_1[n]$



Dashed blue line = DTFT

Figure 3: DFT corresponding to $x_2[n]$



Dashed blue line = DTFT

Figure 4: DFT corresponding to $x_3[n]$

$|X(k)|$ $\angle X(k)$

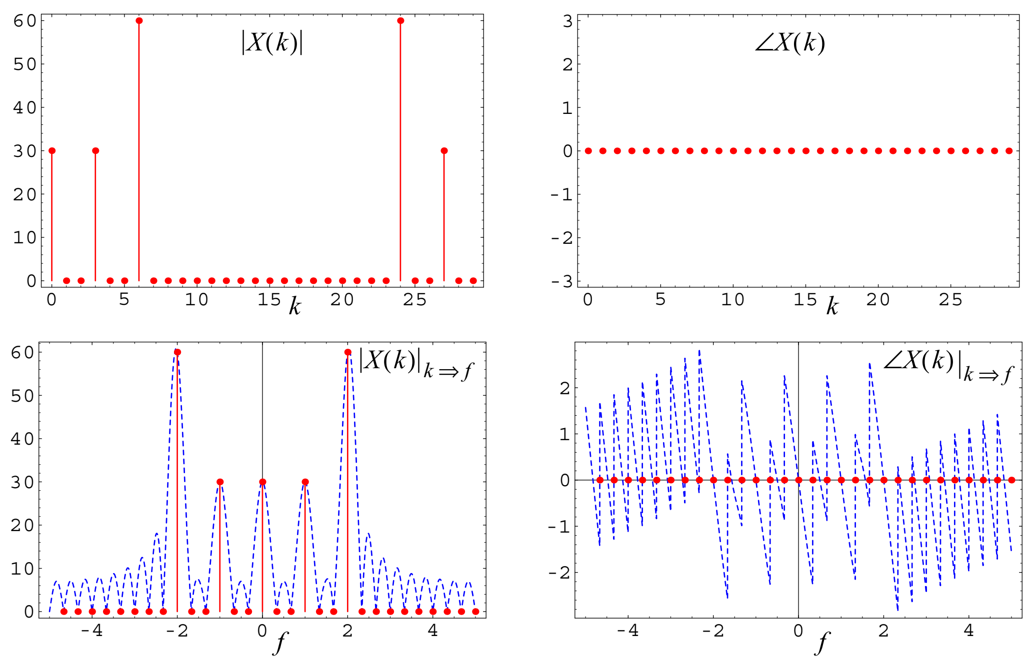
k k

$|X(k)|_{k \Rightarrow f}$ $\angle X(k)|_{k \Rightarrow f}$

f f

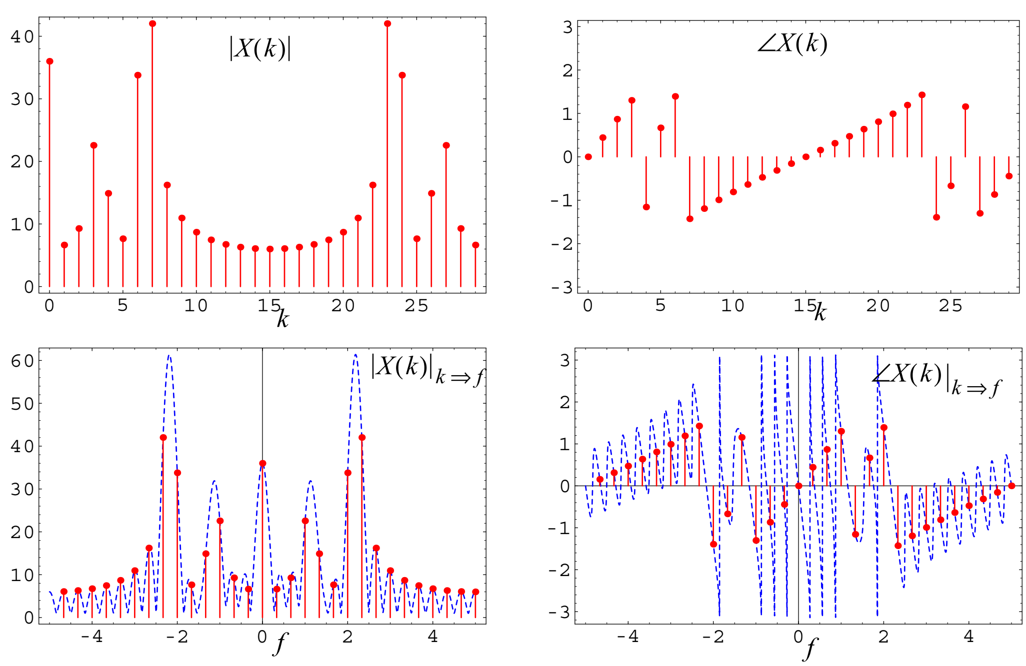
Dashed blue line = DTFT

Figure 5: DFT corresponding to $x_4[n]$



Dashed blue line = DTFT

Figure 6: DFT corresponding to $x_5[n]$



Dashed blue line = DTFT

Figure 7: DFT corresponding to $x_6[n]$