Spectral leakage and windowing

1. Introduction

In these notes, we introduce the idea of *windowing* for reducing the effects of *spectral leakage*, and illustrated this concept with a couple of simple examples.

2. Spectral leakage and windowing

A. Introduction

Previously, we saw that the DFT samples the DTFT at discrete frequencies; we also observed that when a signal consists only of frequencies that are integer multiples of f_s/N, where N is the length of the time-domain sequence $x[n]$ (and consequently the length of the DFT $X(k)$), those frequencies have nonzero coefficients, while all other coefficients are zero (see Figures 2, 3 and 6 in the *Discrete Fourier Transform* notes). However, when a signal consists of frequencies that are *not* integer multiples of f_s/N, the DFT no longer samples the DTFT at its peaks and zero crossings, and therefore, the DFT coefficients are no longer zero for frequencies not in the original infinite-length signal (see Figures 4, 5 and 7 in the *Discrete Fourier Transform* notes). In fact, the frequency content is spread out (i.e. nonzero) over the full range of DFT coefficients $X(k)$. As in the case of the DTFT, this *spectral leakage* is caused by *finite-length* sampling that occurs for any practical application. Increasing the sampling frequency, thereby generating longer discrete-time sequences for equivalent sampling times, reduces spectral leakage, but does not eliminate the problem.

One popular method for mitigating spectral leakage in the DFT estimation of spectral content is called *data windowing*. In this method, the original signal $x[n]$, $n \in \{0, 1, \ldots, N-1\}$, is modified by multiplication with a windowing function that approaches zero near $n = 0$ and $n = N-1$, and reaches a peak of one near $n = N/2$. While there are many possible choices of windowing functions, one popular choice is the *Hamming window function* $h[n]$, which is given below:

$$h[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \; n \in \{0, 1, \ldots, N-1\}. \tag{1}$$

The role of data windowing is to reduce the artificial high frequencies introduced in the DFT by finite-length sampling. This is perhaps best seen through a couple of examples.

B. Example #1

Here, we consider the sampled sequence $x_1[n]$ given by,

$$x_1[n] = \begin{cases} x_c(n/f_s) & 0 \le n < N \\ 0 & elsewhere \end{cases} \tag{2}$$

where,

$$x_c(t) = \cos[2\pi(2 + 1/6)t], \tag{3}$$

$f_s = 20$ Hz, and $N = 60$; that is, $x_c(t)$ is sampled for 3 seconds at 20Hz. Figure 1 below plots $x_1[n]$, $h[n]$ and $x_1[n] \bullet h[n]$, where the $\bullet$ operator indicates element by element multiplication of the vectors $x_1[n]$ and $h[n]$. Each one of these sequences is purposefully plotted for a range of n outside the range $n \in \{0, 1, \ldots, N-1\}$, to illustrate the effect of sampling for only a finite length of time. Conceptually, observe that the finite-length sampling process implicitly assumes the signal $x_1[n]$ is zero for values of $n < 0$ and $n \ge N$; this is equivalent to multiplying the original continuous-time signal $x_c(t)$ by a pulse of width three. This implicit time limiting causes the crisp frequency content of the original cosine function (at $\pm(2 + 1/6)$ Hz) to leak to other frequencies throughout the frequency spectrum. Observe for example, the sharp transition from zero to nonzero at $n = 0$ and $n = N-1$. The frequency content of such a transition, while not present in the original continuous-time signal, is a part of the finite-length sampled sequence.

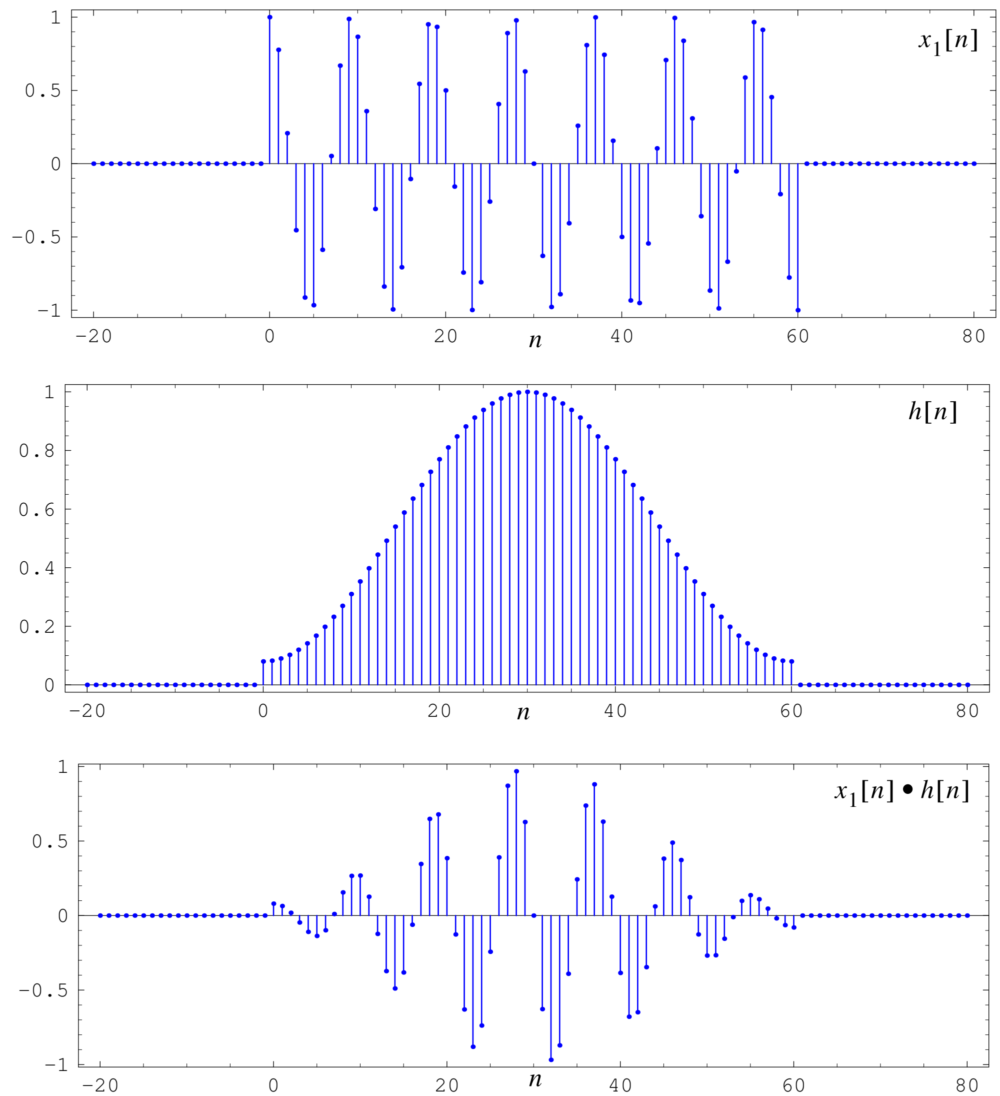

Figure 1

Note, however, that when we multiply the sampled sequence $x_1[n]$ by the Hamming window function $h[n]$, the sharp transitions at $n = 0$ and $n = N-1$ are substantially reduced. In other words, we have replaced multiplication by a square pulse with a smoother, tapered pulse instead (i.e. the Hamming window).

Let us now compare the DTFT and DFT for the original sampled sequence $x_1[n]$ and the windowed sampled sequence $x_1[n] \bullet h[n]$. In Figure 2, we plot the magnitude DTFT and DFT, for the sequences $x_1[n]$ and $x_1[n] \bullet h[n]$. Note that the DTFT and DFT for the windowed sequence exhibits significantly less spectral leakage than the original signal; specifically, spectral leakage is now confined to frequencies around the dominant frequencies of the original continuous-time signal.

C. Example #2

Here, we consider a slightly more complex sampled sequence $x_2[n]$ given by,

$$x_2[n] = \begin{cases} x_c(n/f_s) & 0 \le n < N \\ 0 & elsewhere \end{cases} \tag{4}$$

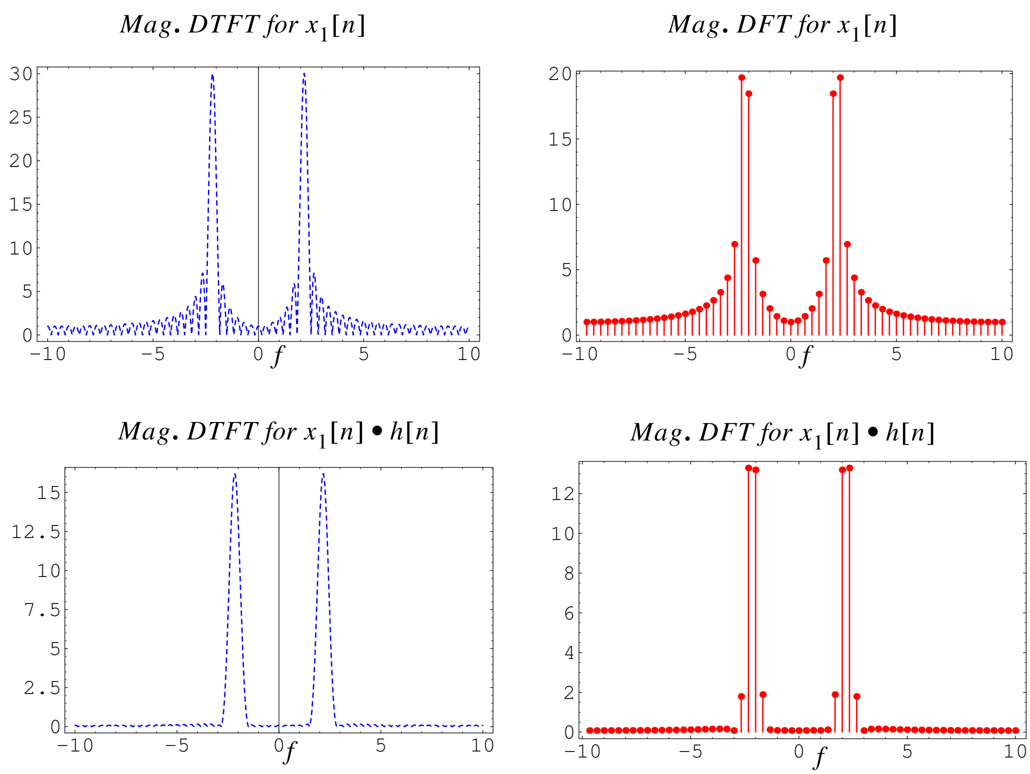

Figure 2

where,

$$x_c(t) = 1 + 2\cos[2\pi(1 + 1/6)t] + 4\cos[2\pi(2 + 1/6)t], \tag{5}$$

$f_s = 20$ Hz, and $N = 60$; that is, $x_c(t)$ is again sampled for 3 seconds at 20Hz. Figure 3 below plots $x_2[n]$, $h[n]$ and $x_2[n] \bullet h[n]$. As before, each one of these sequences is purposefully plotted for a range of n outside the range $n \in \{0, 1, \ldots, N-1\}$, to illustrate the effect of sampling for only a finite length of time. Again note that the implicit time limiting of finite-length sampling causes the crisp frequency content of the original continuous-time function (at 0Hz, $\pm(1 + 1/6)$ Hz and $\pm(2 + 1/6)$ Hz) to leak to other frequencies throughout the frequency spectrum.

Let us now compare the DTFT and DFT for the original sampled sequence $x_2[n]$ and the windowed sampled sequence $x_2[n] \bullet h[n]$. In Figure 4, we plot the magnitude DTFT and DFT, for the sequences $x_2[n]$ and $x_2[n] \bullet h[n]$. Note that the DTFT and DFT for the windowed sequence exhibits significantly less spectral leakage than the original signal; specifically, spectral leakage is now confined to frequencies around the dominant frequencies of the original continuous-time signal.

The discussion above is certainly not comprehensive in its discussion of windowing methods; it is simply intended to introduce basic concepts in windowing for spectrum estimation.

3. Conclusion

The *Mathematica* notebook “spectral_leakage.nb” was used to generate the examples in this set of notes. In subsequent notes, we will look at how some of the ideas in these and previous notes can be applied to the frequency analysis of signals where the frequency content varies over time (e.g. music, speech, etc.).

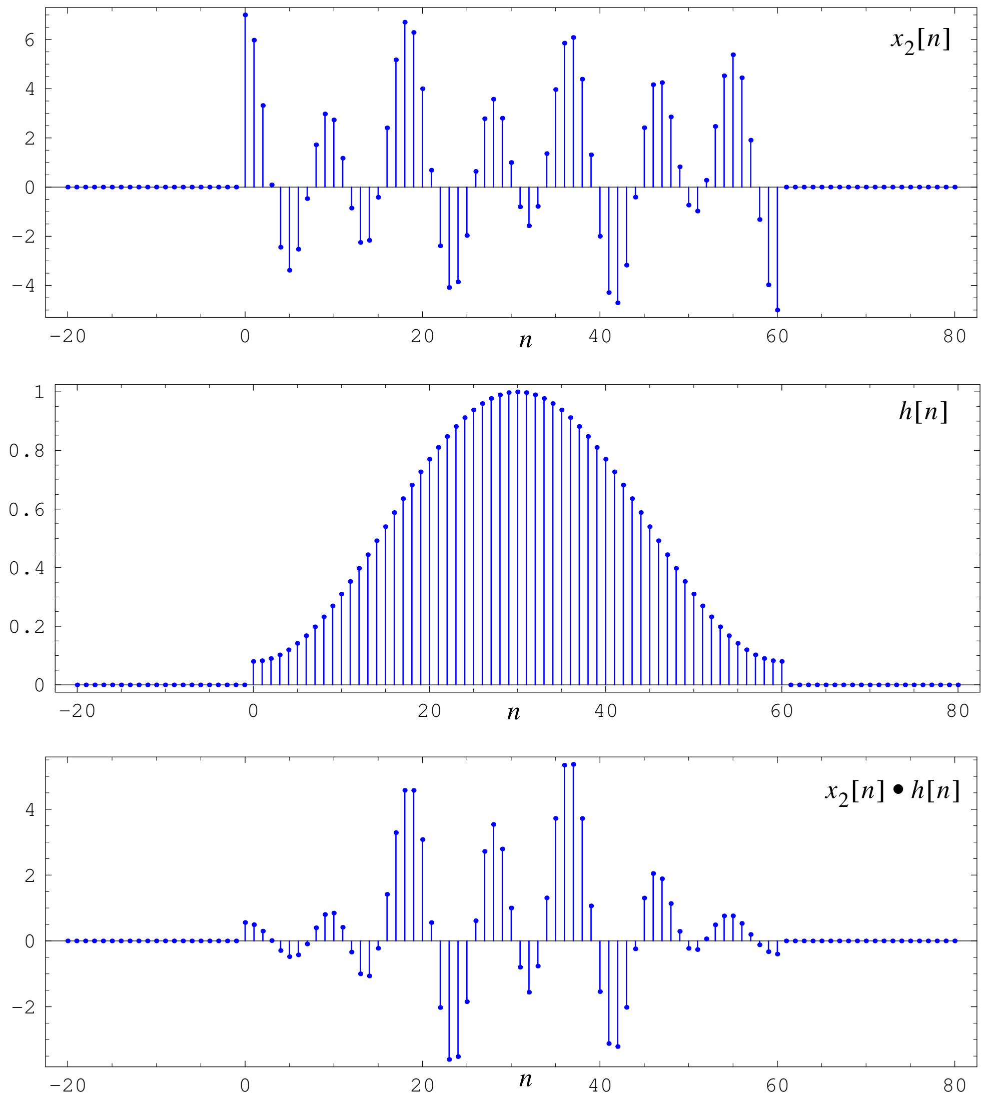

Figure 3

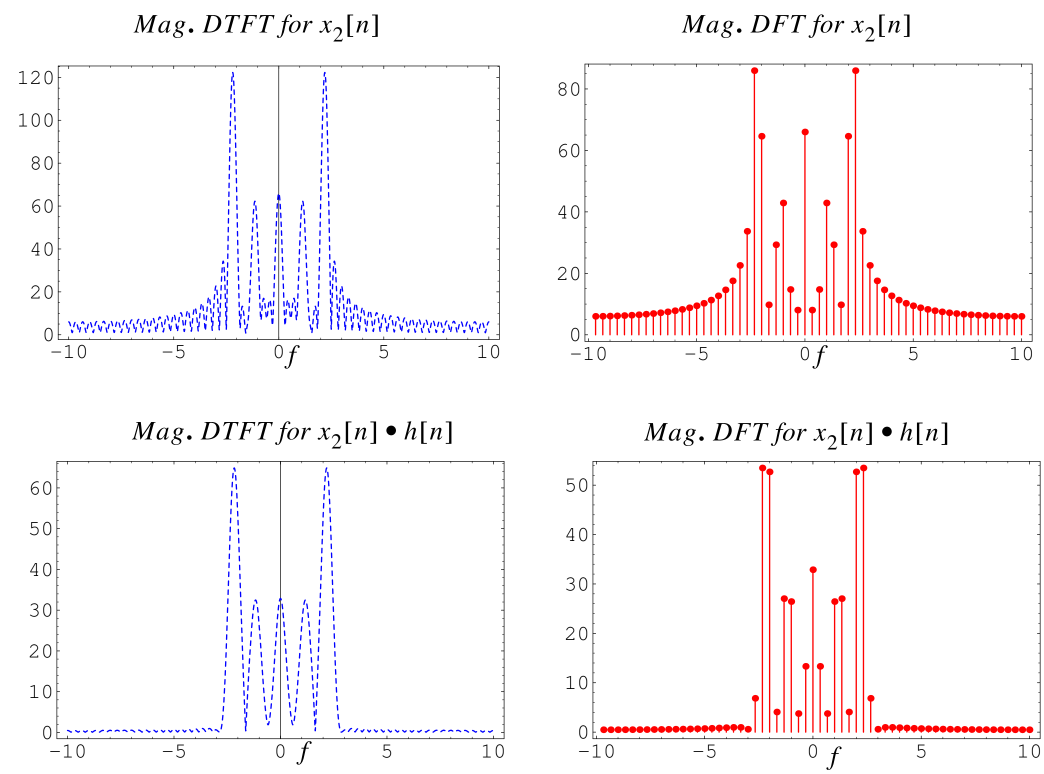
Mag. DTFT for x2[n]
Mag. DFT for x2[n]
Mag. DTFT for x2[n] • h[n]
Mag. DFT for x2[n] • h[n]
f

Figure 4