FIR Filter Design: Part II

1. Introduction

In this set of notes, we consider how we might go about designing FIR filters with arbitrary frequency responses, through composition of multiple single-peak general filters.

2. General FIR filter

A. Introduction

Consider the frequency response $H_g(e^{\mathbf{j}\theta})$ plotted in Figure 1 below. This filter passes through all normalized frequencies θ without modification, but amplifies ($a > 1$) or attenuates ($a < 1$) frequencies in the range $\theta \in [\theta_{c_1}, \theta_{c_2}]$; also, there is a finite-width transition region of width Δ from $H_g(e^{\mathbf{j}\theta}) = 1$ to $H_g(e^{\mathbf{j}\theta}) = a$. As we will show below, assuming that we know the impulse response $h_g[n]$ of this filter in terms of a, θ_{c_1}, θ_{c_2} and Δ, we can easily construct filters that have more complex frequency responses.

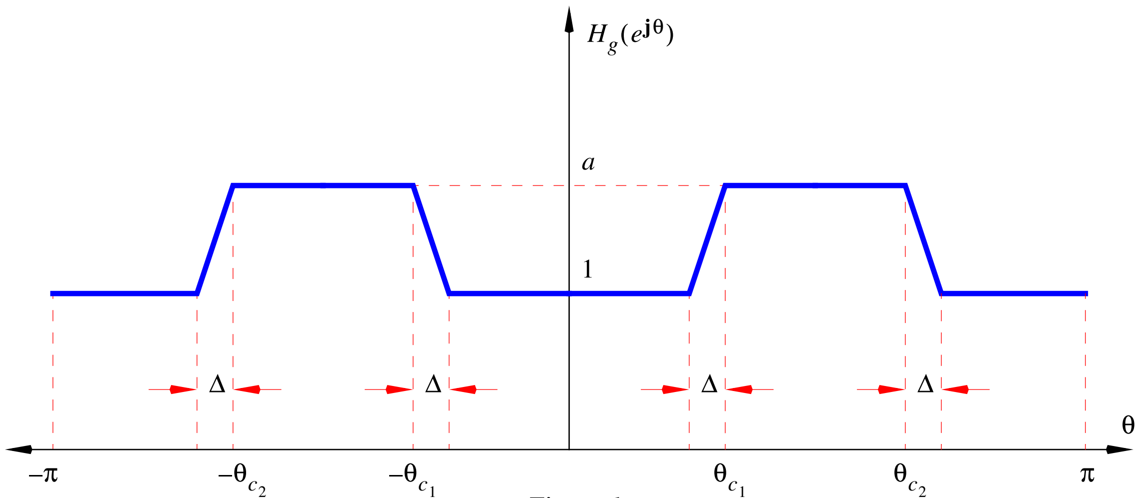


Figure 1

B. Derivation of infinite impulse response

Below, we will use the inverse DTFT to determine the time-domain impulse response $h[n]$ of the above filter, assuming no delay. For this filter, the inverse DTFT is given by,

$$h_g[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_g(e^{\mathbf{j}\theta})e^{\mathbf{j}n\theta}d\theta \tag{1}$$

$$\begin{aligned}
h_g[n] = {} & \frac{1}{2\pi}\int_{-\pi}^{-(\theta_{c_2}+\Delta)} e^{\mathbf{j}n\theta}d\theta + \frac{1}{2\pi}\int_{-(\theta_{c_2}+\Delta)}^{-\theta_{c_2}} \left[\frac{a-1}{\Delta}(\theta+\theta_{c_2})+a\right]e^{\mathbf{j}n\theta}d\theta + \\
& \frac{1}{2\pi}\int_{-\theta_{c_2}}^{-\theta_{c_1}} a e^{\mathbf{j}n\theta}d\theta + \frac{1}{2\pi}\int_{-\theta_{c_1}}^{-(\theta_{c_1}-\Delta)} \left[\frac{1-a}{\Delta}(\theta+\theta_{c_1})+a\right]e^{\mathbf{j}n\theta}d\theta + \\
& \frac{1}{2\pi}\int_{-(\theta_{c_1}-\Delta)}^{(\theta_{c_1}-\Delta)} e^{\mathbf{j}n\theta}d\theta + \\
& \frac{1}{2\pi}\int_{(\theta_{c_1}-\Delta)}^{\theta_{c_1}} \left[\frac{a-1}{\Delta}(\theta-\theta_{c_1})+a\right]e^{\mathbf{j}n\theta}d\theta + \frac{1}{2\pi}\int_{\theta_{c_1}}^{\theta_{c_2}} a e^{\mathbf{j}n\theta}d\theta + \\
& \frac{1}{2\pi}\int_{\theta_{c_2}}^{(\theta_{c_2}+\Delta)} \left[\frac{1-a}{\Delta}(\theta-\theta_{c_2})+a\right]e^{\mathbf{j}n\theta}d\theta + \frac{1}{2\pi}\int_{(\theta_{c_2}+\Delta)}^{\pi} e^{\mathbf{j}n\theta}d\theta
\end{aligned} \tag{2}$$

The integral in equation (2) is tedious to compute and simplify; therefore, we solve the integration using *Mathematica* (see “fir_filter_design_part2.nb”) and arrive at the following expression for $h[n]$:

$$h_g[n] = (1-a)\left[\frac{\cos(n(\theta_{c_1}-\Delta)) + \cos(n(\theta_{c_2}+\Delta)) - \cos(n\theta_{c_1}) - \cos(n\theta_{c_2})}{n^2\pi\Delta}\right] + \frac{\sin(n\pi)}{n\pi},$$
$$-\infty < n < \infty. \quad (3)$$

Note that equation (3) can be further simplified, since,

$$\frac{\sin(n\pi)}{n\pi} = \delta[n] \quad (4)$$

for integer n. Therefore,

$$h_g[n] = (1-a)\left[\frac{\cos(n(\theta_{c_1}-\Delta)) + \cos(n(\theta_{c_2}+\Delta)) - \cos(n\theta_{c_1}) - \cos(n\theta_{c_2})}{n^2\pi\Delta}\right] + \delta[n], -\infty < n < \infty. \quad (5)$$

Note that, although $h_g[n]$ is infinite in length, both forwards and backwards in time, the impulse response does decay to zero as $|n| \to \infty$. Figure 2 below, for example, plots $h_g[n]$, $-50 \le n \le 50$, for $\theta_{c_1} = 1$, $\theta_{c_2} = 3/2$, $\Delta = 1/10$ and $a = 2$.

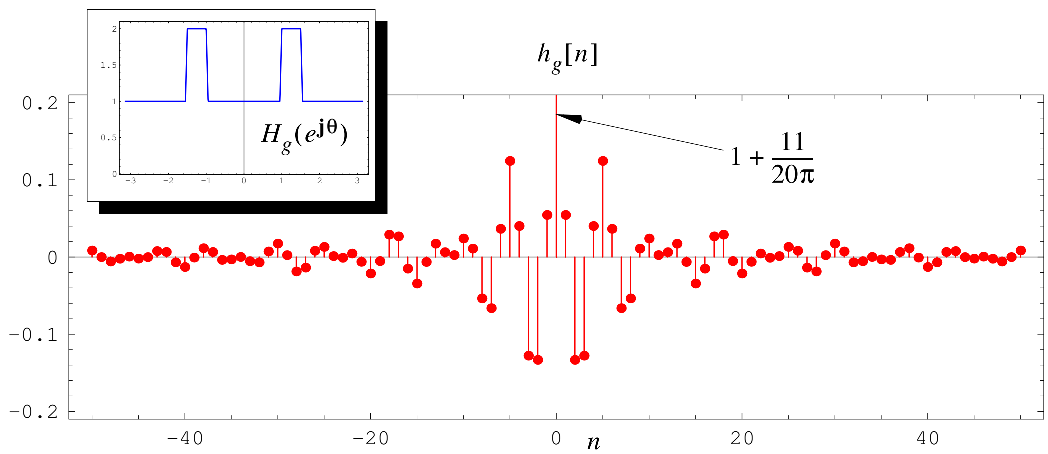


Figure 2

C. Approximating of infinite impulse response with a causal FIR filter

Now, we will try to approximate the infinite impulse response in equation (5) with a causal FIR filter using the same procedure as we have used before: First, we will retain values for $h_g[n]$ only for a limited range of n,

$$-n_{max} \le n \le n_{max} \quad (6)$$

since values of $h_g[n]$ approach zero as $|n| \to \infty$. Let us denote this finite impulse response as $h_{g1}[n]$, such that,

$$h_{g1}[n] = \begin{cases} h_g[n] & -n_{max} \le n \le n_{max} \\ 0 & elsewhere \end{cases} \quad (7)$$

Second, we will shift $h_{g1}[n]$ to be causal; let us denote the resulting impulse response as $\hat{h}_g[n]$ such that,

$$\hat{h}_g[n] = h_{g1}[n - n_{max}]. \quad (8)$$

Note that this shift will not change the steady-state frequency response of the resulting system, except to introduce a delay at the output. Thus, the difference equation corresponding to $\hat{h}_g[n]$ can be written as:

$$y[n] = \sum_{k=0}^{2n_{max}} b_k x[n-k] \tag{9}$$

where,

$$b_k = \hat{h}_g[n] = h_g[-n_{max}+k] \text{, } k \in \{0, 1, \ldots, 2n_{max}\} . \tag{10}$$

3. Composite filter example

A. Introduction

Given that we have computed the impulse response $h_g[n]$ for the filter in Figure 1, we are now in position to construct filters with more complex frequency responses. Suppose, for example, we wanted to construct a filter with the idealized frequency response $H_c(e^{\mathbf{j}\theta})$ plotted in Figure 3 below for $\theta \in [0, \pi]$. (Since $H(e^{\mathbf{j}\theta})$ is an even function, we do not need to explicitly plot it for $\theta = [-\pi, 0)$.)

We could, of course, derive the impulse response $h_c[n]$ of this filter by breaking up the inverse DTFT into piecewise-continuous integrals (as we did, for example, in equation (2) above) and solving the resulting expression. Given that we have already compute $h_g[n]$ [equation (5)], this difficult and tedious procedure is not necessary, however. Instead, we can treat $H_c(e^{\mathbf{j}\theta})$ as a composite filter consisting of two different $H_g(e^{\mathbf{j}\theta})$ filters connected in series.

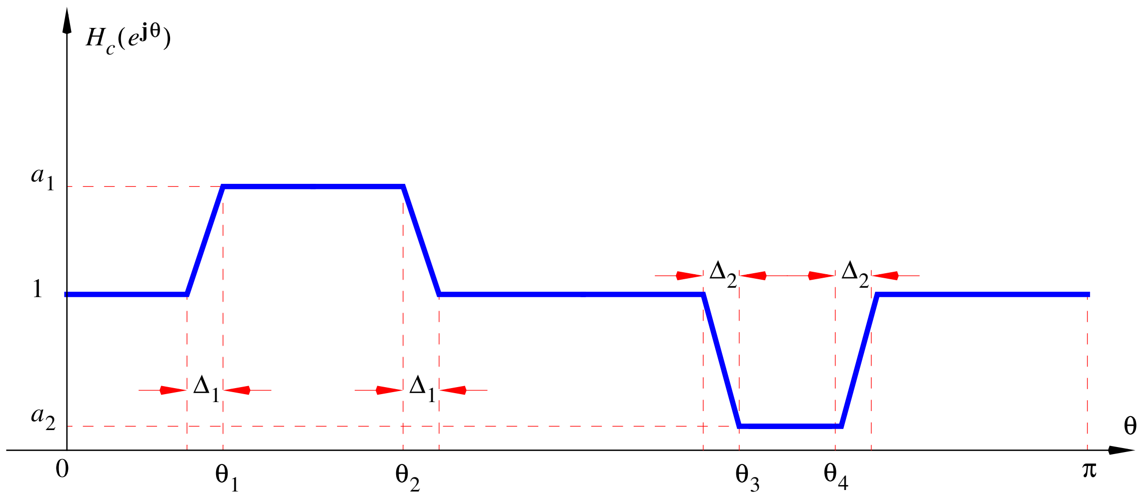


Figure 3

Let,

$$H_1(e^{\mathbf{j}\theta}) = H_g(e^{\mathbf{j}\theta})\Big|_{a \to a_1,\, \theta_{c1} \to \theta_1,\, \theta_{c2} \to \theta_2,\, \Delta \to \Delta_1} \tag{11}$$

and

$$H_2(e^{\mathbf{j}\theta}) = H_g(e^{\mathbf{j}\theta})\Big|_{a \to a_2,\, \theta_{c1} \to \theta_3,\, \theta_{c2} \to \theta_4,\, \Delta \to \Delta_2} \tag{12}$$

where $H_g(e^{\mathbf{j}\theta})$ denotes the filter in Figure 1. Then, we can represent $H_c(e^{\mathbf{j}\theta})$ in Figure 3 as,

$$H_c(e^{\mathbf{j}\theta}) = H_1(e^{\mathbf{j}\theta})H_2(e^{\mathbf{j}\theta}) \tag{13}$$

or, in the time domain as,

$$h_c[n] \approx \hat{h}_1[n] * \hat{h}_2[n] \tag{14}$$

where $\hat{h}_1[n]$ and $\hat{h}_2[n]$ denote the causal FIR approximations of the infinite impulse response of filters $H_1(e^{\mathbf{j}\theta})$ and $H_2(e^{\mathbf{j}\theta})$, respectively. Below, we give one example of such a composite filter.

B. Example

Let us assume that we want to filter a piece of music, sampled at $f_s = 32$ kHz, with the composite filter shown in Figure 3 and the following numeric values:

$$a_1 = 10^{(12/20)} \approx 3.981 \text{ . } f_1 = 375 \text{ Hz}, f_2 = 1500 \text{ Hz}, \tag{15}$$

$$a_2 = 10^{(-12/20)} \approx 0.2512 \text{ , } f_3 = 6 \text{ kHz}, f_4 = 15 \text{ kHz}, \tag{16}$$

and $\Delta_f = 100$ Hz for both segments. Note that the value of a_1 corresponds to +12dB (decibels), while the value of a_2 corresponds to -12dB.[1] First, we need to convert the frequency values above to normalized frequency values using the conversion formula:

$$\theta = \frac{2\pi f}{f_s} \tag{17}$$

Applying equation (17), we get the following values in terms of normalized frequency for $f_s = 32$ kHz:

$$\theta_1 = 3\pi/128 \text{ , } \theta_2 = 3\pi/32 \text{ , } \theta_3 = 3\pi/8 \text{ , } \theta_4 = 15\pi/16 \text{ and } \Delta_1 = \Delta_2 = 2\pi\Delta_f/f_s = \pi/160 \text{ . } \tag{18}$$

The composite filter transfer function $H_c(e^{\mathbf{j}\theta})$ for the numeric values in (18) is plotted in Figure 4 below, in terms of normal amplitude, and decibel amplitude.

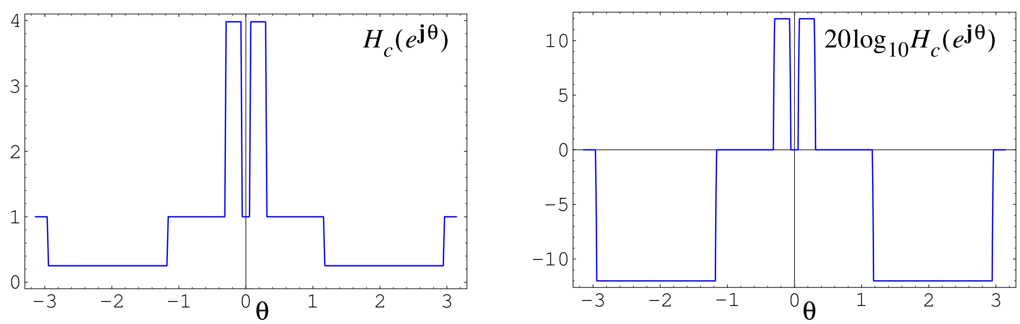


Figure 4

The FIR impulse-function approximations $\hat{h}_1[n]$ and $\hat{h}_2[n]$, corresponding to the filters defined by (11) and (12), respectively, are plotted in Figure 5 below for $n_{max} = 500$. Note that n_{max} was chosen such that the $h[n]$ coefficients near $n = 0$ and $n = 2n_{max}$ are very close to zero in value. Now, we compute $h_c[n]$, the composite-filter FIR impulse-function approximation, by convolving $\hat{h}_1[n]$ and $\hat{h}_2[n]$ [see equation (14) above]; $h_c[n]$ is plotted in Figure 6.

Observe from Figure 6, that $h_c[n]$ is approximately twice as long as $\hat{h}_1[n]$' and $\hat{h}_2[n]$ (2001 *vs.* 1001) as should be expected from the convolution operator. Also note that many of the $h_c[n]$ coefficients are very close to zero in value; hence, we can derive a shorter FIR filter by retaining only $h_c[n]$ values in the range $n \in [1001 - n_2, 1001 + n_2]$, and shifting the resulting impulse response to start at $n = 0$. From Figure 6, we suggest that a value of $n_2 = 300$ seems appropriate since $h_c[n]$ values for $n < 700$ and $n > 1300$ appear negligibly small. Let us denote the shortened FIR filter as $\hat{h}_c[n]$:

$$\hat{h}_c[n] = h_c[n + 1001 - n_2] \text{ , } n \in \{0, 1, \ldots, 2n_2\} \text{ , } n_2 = 300 \text{ .} \tag{19}$$

1. *For an amplitude* A*, the corresponding dB value is given by* $20\log_{10}A$.

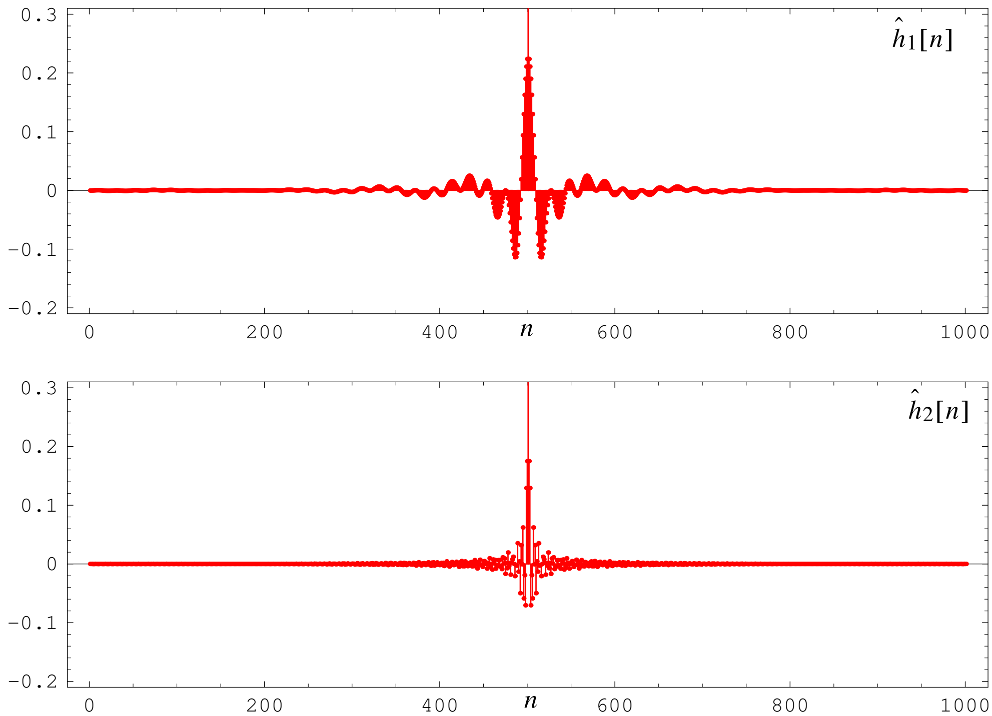


Figure 5

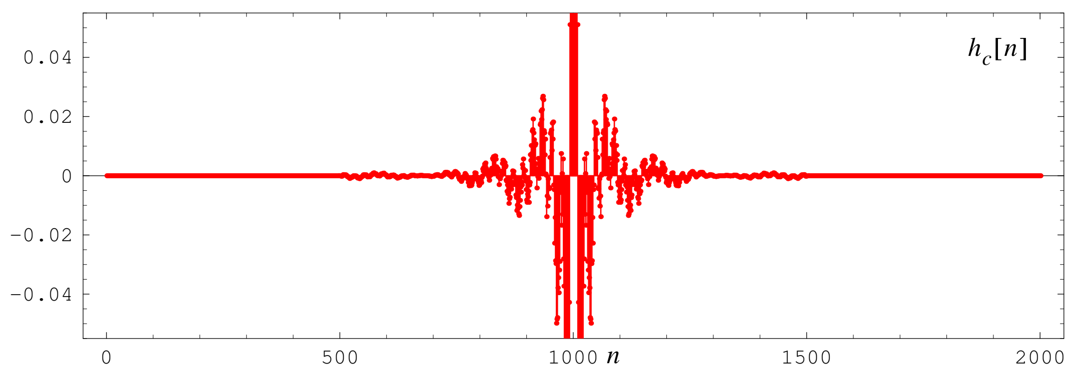


Figure 6

This impulse response ($\hat{h}_c[n]$) is plotted in Figure 7 below. Therefore, the difference equation corresponding to this impulse function is given by,

$$y[n] = \sum_{k=0}^{2n_2} b_k x[n-k] \tag{20}$$

where,

$$b_k = \hat{h}_c[k], \quad k \in \{0, 1, \ldots, 2n_2\} \quad , n_2 = 300\,. \tag{21}$$

In Figure 8, we plot the magnitude frequency response $\hat{H}_c(e^{j\theta})$ of system (20) given by,

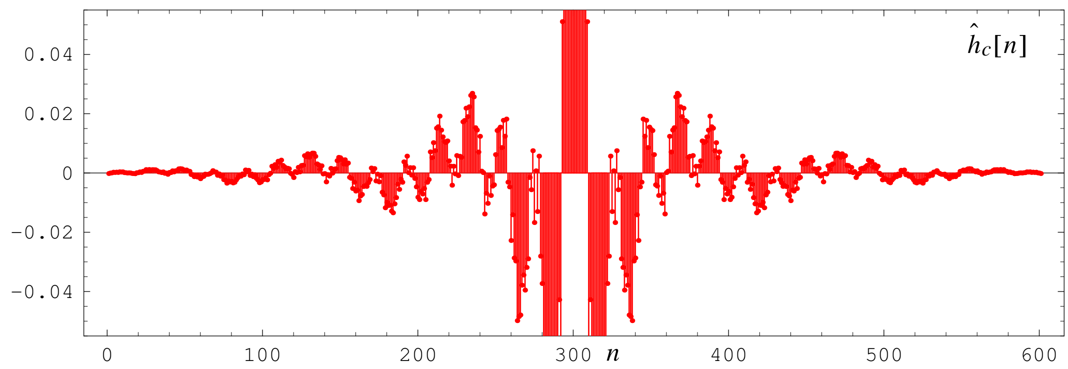


Figure 7

$$\hat{H}_c(e^{\mathbf{j}\theta}) = \sum_{n=0}^{600} \hat{h}_c[n]e^{-\mathbf{j}n\theta} \tag{22}$$

both as a function of normalized frequency θ and real frequency f. Note how closely we are able to approximate our desired frequency response (see Figure 4) with this FIR filter of length 601; also note that we never had to compute $h_c[n]$ using the inverse DTFT, but rather computed it through time-domain convolution of $\hat{h}_1[n]$ and $\hat{h}_2[n]$.

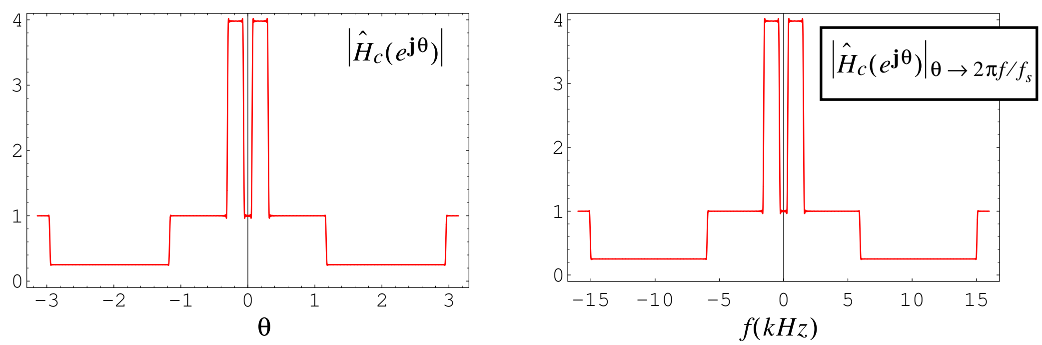


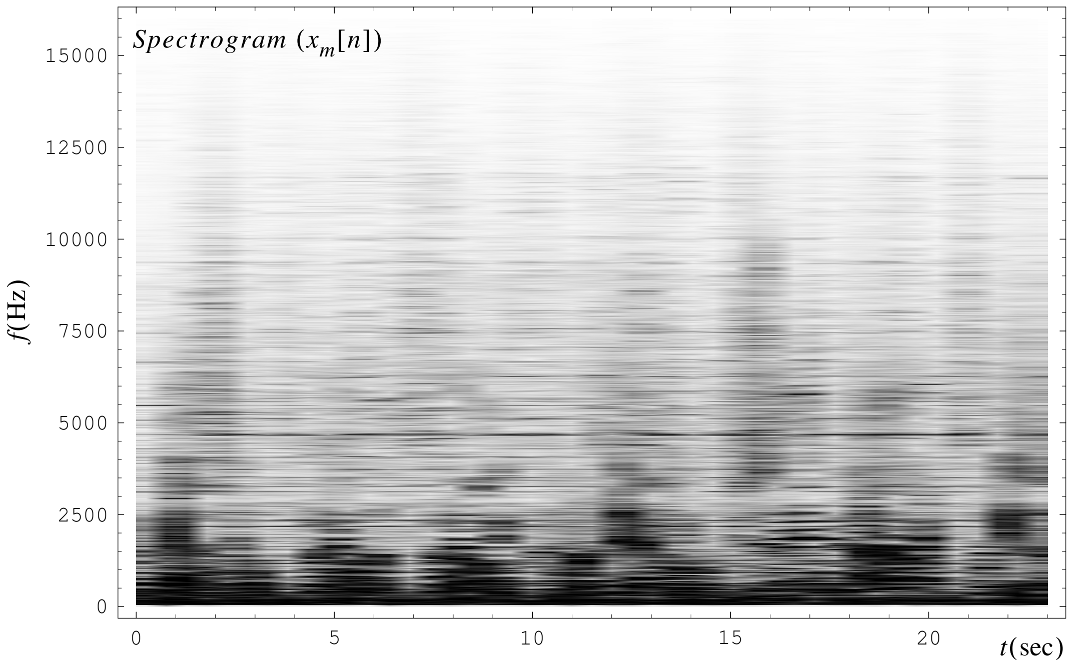
Figure 8

To see this filter in action, we convolve $\hat{h}_c[n]$ with a sampled music file $x_m[n]$ [1], and compare the frequency content of the original music file $x_m[n]$ and the filtered music file $y_m[n]$ where,

$$y_m[n] = x_m[n] * \hat{h}_c[n]. \tag{23}$$

In Figure 9 below, we plot the spectrograms[2] of $x_m[n]$ and $y_m[n]$, where we analyze frequency content in one-second long segments with a 50% overlap. We also plot one column of both spectrograms (the magnitude FFT for one of the one-second segments of $x_m[n]$ and $y_m[n]$) in Figure 10 below. Note, how the frequency content in the range $f \in [375\text{Hz}, 1500\text{Hz}]$ has been amplified, while the frequency content in the range $f \in [6\text{kHz}, 15\text{kHz}]$ has been significantly attenuated, as we should expect from the frequency response of $\hat{h}_c[n]$.

1. *See the web page for this music file — a 23.06 second segment of Kenny Roger's "The Gambler," sampled at $f_s = 32$ kHz.*
2. *Recall from previous notes that the spectrogram gives us the magnitude frequency content of a signal for short segments of a long discrete-time sequence.*


Spectrogram ($x_m[n]$)
15000
12500
10000
7500
5000
2500
0
f(Hz)
0
5
10
15
20
t(sec)

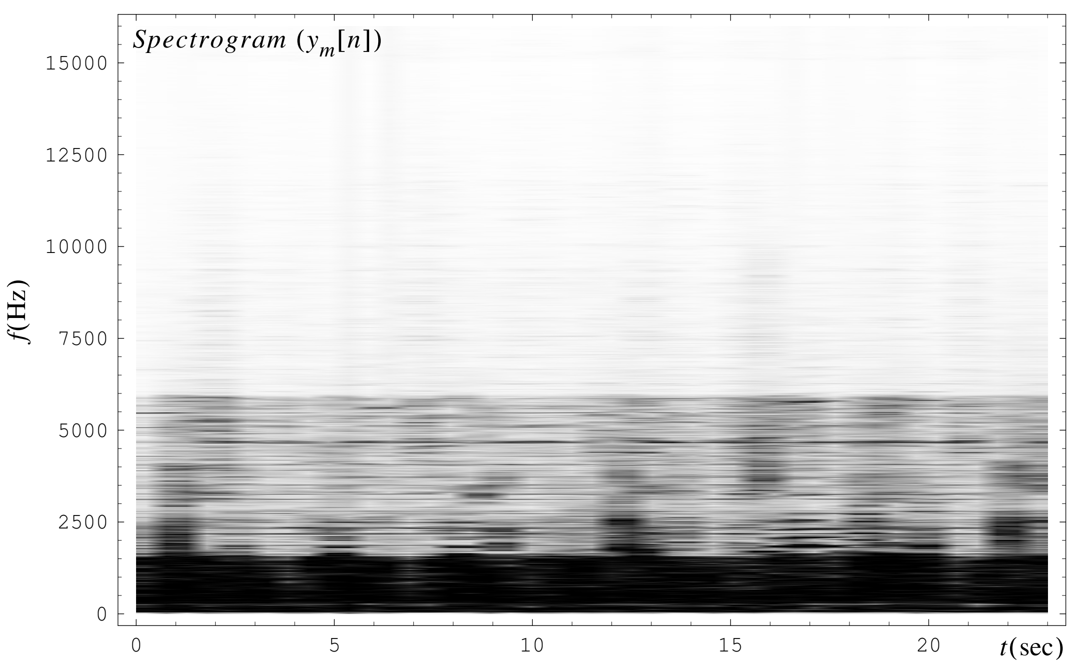

Spectrogram ($y_m[n]$)
15000
12500
10000
7500
5000
2500
0
f(Hz)
0
5
10
15
20
t(sec)

Figure 9

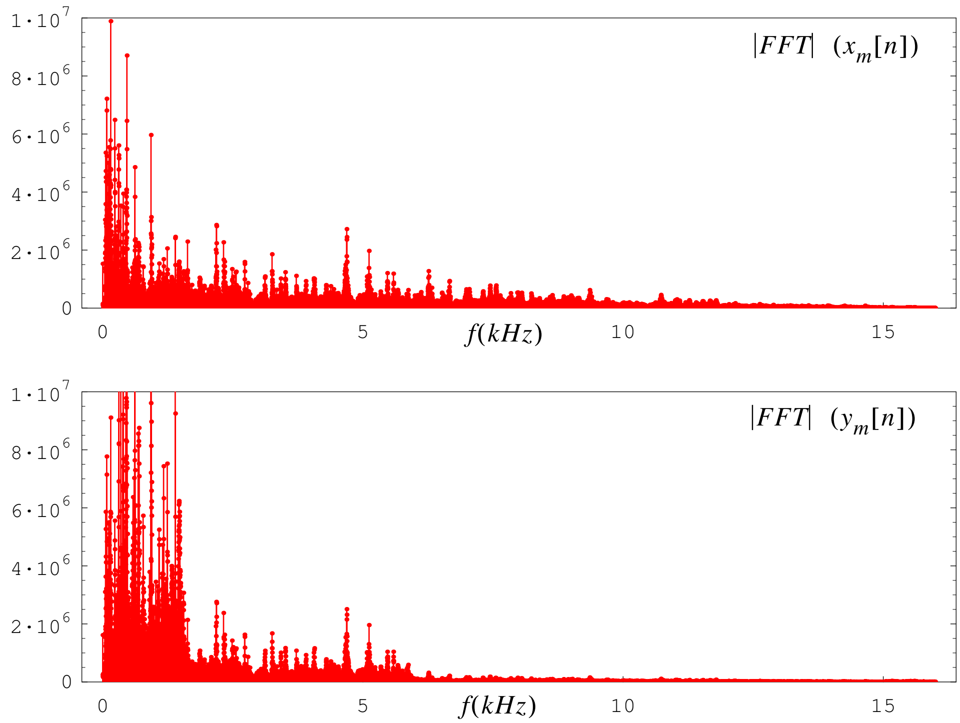


Figure 10

4. Conclusion

The *Mathematica* notebook “fir_filter_design_part2.nb” was used to generate the detailed filtering example in this set of notes; that notebook also contains a second composite filter example for the same piece of music, where all the design values (i.e. θ_1, θ_2, Δ_1, θ_3, θ_4 and Δ_2) are the same, except the values for a_1 and a_2 are reversed; that is,

$$a_1 = 10^{(-12/20)} \text{ and } a_2 = 10^{(12/20)}. \tag{24}$$

The filtered and unfiltered music segments referred to in these notes is available in mp3 and wav formats on the course web site.