Introduction to IIR systems

1. Introduction

So far, we have extensively explored the analysis and design of FIR filters. In this set of notes, we consider the analysis of IIR (infinite-impulse response) filters. Specifically, we will address the following three important questions concerning IIR filters:

1. How do we compute the impulse response of IIR filters?
2. How do we analyze the stability of IIR filters, something we did not have to consider for FIR filters?
3. How do we compute the frequency response of an IIR filter?

All of the above questions will be solved with a new transform, namely the z-transform. As we will see shortly, the z-transform is a generalization of the DTFT, and is analogous to the Laplace transform for differential equations.

Given that we have to worry about the stability of IIR systems and that such systems are more difficult to analyze, we might wonder why we bother with these systems at all. IIR systems are worthy of study for at least three important reasons: (1) as discrete-time approximations of differential equations for computer simulation; (2) the analysis of discrete-time, or computer-in-the-loop feedback control systems; and (3) in the design of shorter filters with frequency-response characteristics similar to those of longer FIR filters.

2. The z-transform

A. Introduction

In this section, we introduce a new transform, the z-transform, which will allow us to analyze IIR systems of the following form:

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k] \tag{1}$$

where some of the a_l coefficients are nonzero constants. We have already seen some simple examples of IIR (or recursive) systems in the 1/24 and 1/29 lecture notes. The z-transform will allow us to analyze a more general class of IIR systems by converting difference equations to polynomial equations.

B. Definition

The *bilateral z-transform* $X(z)$ of a discrete-time sequence $x[n]$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{2}$$

If we compare the definition of $X(z)$ to the definition of the DTFT $X(e^{\mathbf{j}\theta})$,

$$X(e^{\mathbf{j}\theta}) = \sum_{n=-\infty}^{\infty} x[n] e^{-\mathbf{j}n\theta} \tag{3}$$

we can derive a relationship between the z-transform and the DTFT:

$$X(e^{\mathbf{j}\theta}) = X(z)\big|_{z = e^{\mathbf{j}\theta}}. \tag{4}$$

Below, we derive some important transform pairs that associate time-domain signals with their respective z-transform; we will use these transform pairs later in our analysis of IIR systems.

C. Transform pairs

Below, the $\Leftrightarrow$ symbol denotes a correspondence between the time-domain representation of a signal and the z-domain representation of that same signal.

Linearity:

$$\alpha x_1[n] + \beta x_2[n] \quad \Leftrightarrow \quad \alpha X_1(z) + \beta X_2(z) \tag{5}$$

Proof: Let,

$$w[n] = \alpha x_1[n] + \beta x_2[n]. \tag{6}$$

Then, by definition (2),

$$W(z) = \sum_{n=-\infty}^{\infty} w[n] z^{-n} \tag{7}$$

$$W(z) = \sum_{n=-\infty}^{\infty} (\alpha x_1[n] + \beta x_2[n]) z^{-n} \tag{8}$$

$$W(z) = \sum_{n=-\infty}^{\infty} \alpha x_1[n] z^{-n} + \beta x_2[n] z^{-n} \tag{9}$$

$$W(z) = \alpha\left(\sum_{n=-\infty}^{\infty} x_1[n] z^{-n}\right) + \beta\left(\sum_{n=-\infty}^{\infty} x_2[n] z^{-n}\right) = \alpha X_1(z) + \beta X_2(z). \tag{10}$$

Time-shift:

$$x[n-n_0] \quad \Leftrightarrow \quad z^{-n_0} X(z) \tag{11}$$

Proof: Let,

$$w[n] = x[n-n_0] \tag{12}$$

Then, by definition (2),

$$W(z) = \sum_{n=-\infty}^{\infty} w[n] z^{-n} \tag{13}$$

$$W(z) = \sum_{n=-\infty}^{\infty} x[n-n_0] z^{-n} \tag{14}$$

Let us make the substitution $k = n - n_0$ into equation (14):

$$W(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-(k+n_0)} \tag{15}$$

$$W(z) = z^{-n_0}\left(\sum_{k=-\infty}^{\infty} x[k] z^{-k}\right) = z^{-n_0} X(z). \tag{16}$$

Convolution:

$$x_1[n] * x_2[n] \quad \Leftrightarrow \quad X_1(z) X_2(z) \tag{17}$$

Proof: Let,

$$w[n] = x_1[n] * x_2[n] \tag{18}$$

Then, by definition (2),

$$W(z) = \sum_{n=-\infty}^{\infty} w[n]z^{-n} \tag{19}$$

$$W(z) = \sum_{n=-\infty}^{\infty} (x_1[n] * x_2[n])z^{-n} \tag{20}$$

$$W(z) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right) z^{-n} \tag{21}$$

Let us make the substitution $m = n - k$ into equation (21):

$$W(z) = \sum_{m=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_1[k]x_2[m] \right) z^{-(m+k)} \tag{22}$$

$$W(z) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (x_1[k]z^{-k})(x_2[m]z^{-m}) \tag{23}$$

$$W(z) = \left(\sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \right) \left(\sum_{m=-\infty}^{\infty} x_2[m]z^{-m} \right) = X_1(z)X_2(z)\,. \tag{24}$$

<u>Discrete-time delta function</u>:

$$\delta[n] \quad \Leftrightarrow \quad 1 \tag{25}$$

Proof: Let,

$$w[n] = \delta[n] \tag{26}$$

Then, by definition (2),

$$W(z) = \sum_{n=-\infty}^{\infty} w[n]z^{-n} \tag{27}$$

$$W(z) = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} = z^{-0} = 1\,. \tag{28}$$

<u>Shifted discrete-time delta function</u>:

$$\delta[n - n_0] \quad \Leftrightarrow \quad z^{-n_0} \tag{29}$$

Proof: We can derive transform pair (29) two ways. First, we can appeal directly to the definition of the z-transform. Let,

$$w[n] = \delta[n - n_0] \tag{30}$$

Then, by definition (2),

$$W(z) = \sum_{n=-\infty}^{\infty} w[n]z^{-n} \tag{31}$$

$$W(z) = \sum_{n=-\infty}^{\infty} \delta[n-n_0]z^{-n} = z^{-n_0}. \tag{32}$$

Alternatively we could combine transform pairs (11) and (25). Let $x[n] = \delta[n]$. Then, by transform pair (25),

$$X(z) = 1. \tag{33}$$

Now we use transform pair (11) to determine the z-transform for $w[n] = x[n-n_0]$:

$$W(z) = z^{-n_0}X(z) = z^{-n_0}. \tag{34}$$

<u>Discrete-time, exponential step input</u>:

$$a^n u[n] \quad \Leftrightarrow \quad \frac{1}{1-az^{-1}},\ |a| < z. \tag{35}$$

Proof: Let,

$$w[n] = a^n u[n] \tag{36}$$

Then, by definition (2),

$$W(z) = \sum_{n=-\infty}^{\infty} (a^n u[n])z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} \tag{37}$$

$$W(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}},\ |az^{-1}| < 1, \tag{38}$$

where, in (38), we have used the infinite geometric sum identity:

$$\sum_{n=0}^{\infty} b^n = \frac{1}{1-b},\ |b| < 1. \tag{39}$$

The condition $|az^{-1}| < 1$ determines the region of convergence for the z-transform, and can be restated as $|a| < z$. Thus,

$$W(z) = \frac{1}{1-az^{-1}},\ |a| < z. \tag{40}$$

Although at this point it may seem odd that we took the time to derive transform pair (35), we will have many occasions to use that particular transform pair in our analysis of IIR systems. The table below summarizes the transform pairs that we derived in this section.

Transform pair label	$w[n] \quad \Leftrightarrow \quad W(z)$
Linearity	$\alpha x_1[n] + \beta x_2[n] \quad \Leftrightarrow \quad \alpha X_1(z) + \beta X_2(z)$
Time-shift	$x[n-n_0] \quad \Leftrightarrow \quad z^{-n_0}X(z)$
Convolution	$x_1[n] * x_2[n] \quad \Leftrightarrow \quad X_1(z)X_2(z)$
Discrete-time delta function:	$\delta[n] \quad \Leftrightarrow \quad 1$

Transform pair label	$w[n] \Leftrightarrow W(z)$
Shifted discrete-time delta function:	$\delta[n-n_0] \Leftrightarrow z^{-n_0}$
Discrete-time, exponential step input:	$a^n u[n] \Leftrightarrow \dfrac{1}{1-az^{-1}}, \ \lvert a \rvert < z$

3. Impulse response of IIR systems

A. Introduction

In this section, we apply the z-transform to derive the impulse response of IIR filters.

B. Derivation

Below, we apply the z-transform to the IIR difference equation:

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k] \tag{41}$$

From the transform pairs derived in the previous section, we consider each term in equation (41) separately:

$$y[n] \Leftrightarrow Y(z) \tag{42}$$

$$\sum_{l=1}^{N} a_l y[n-l] \Leftrightarrow \sum_{l=1}^{N} a_l z^{-l} Y(z) = Y(z) \sum_{l=1}^{N} a_l z^{-l} \text{ (linearity and time-shift properties)} \tag{43}$$

$$\sum_{k=0}^{M} b_k x[n-k] \Leftrightarrow \sum_{k=0}^{M} b_k z^{-k} X(z) = X(z) \sum_{k=0}^{M} b_k z^{-k} \text{ (linearity and time-shift properties)} \tag{44}$$

Combining the results of (42) through (44):

$$Y(z) = Y(z) \sum_{l=1}^{N} a_l z^{-l} + X(z) \sum_{k=0}^{M} b_k z^{-k} \tag{45}$$

$$Y(z)\left(1 - \sum_{l=1}^{N} a_l z^{-l}\right) = X(z) \sum_{k=0}^{M} b_k z^{-k} \tag{46}$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{l=1}^{N} a_l z^{-l}} \tag{47}$$

Equation (47) gives us the z-transform $H(z)$ of the impulse response $h[n]$ of the IIR system in (41):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{l=1}^{N} a_l z^{-l}} \tag{48}$$

$H(z)$ is known as the *transfer function* of the system. So, for any input sequence $x[n]$, whose z-transform is given by $X(z)$, the output of the system $y[n]$, whose z-transform is given by $Y(z)$, can be written in the z-domain as,

$$Y(z) = H(z)X(z) \text{ (convolution property).} \tag{49}$$

As we will show in the following sections, we can recover the time-domain representations of $H(z)$ (i.e. $h[n]$) and $Y(z)$ (i.e. $y[n]$) through algebraic manipulation and the transform pairs derived previously.

C. Example #1

Let us consider the following first-order IIR system:

$$y[n] = ay[n-1] + bx[n]. \tag{50}$$

By direct substitution, we can derive the impulse response $h[n]$ of the system by letting $x[n] = \delta[n]$, and assuming $y[n] = 0$, $n < 0$:

$$h[0] = ay[-1] + b\delta[0] = b \tag{51}$$

$$h[1] = ah[0] + b\delta[1] = ah[0] = ba \tag{52}$$

$$h[2] = ah[1] + b\delta[2] = ah[1] = ba^2 \tag{53}$$

Generalizing for arbitrary n,

$$h[n] = ba^n u[n]. \tag{54}$$

Alternatively, we can derive the impulse response through the z-transform:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{l=1}^{N} a_l z^{-l}} = \frac{b}{(1 - az^{-1})} \tag{55}$$

Applying transform pair (35),

$$h[n] = ba^n u[n]. \tag{56}$$

Note that equations (54) and (56) are equivalent. Next, we will consider a more complicated example.

D. Example #2

Let us consider the following general second-order IIR system:

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1]. \tag{57}$$

For this system, direct substitution will no longer work to reveal a general analytic solution for the impulse response $h[n]$. Therefore, let us try to derive the impulse response $h[n]$ instead through the z-transform:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{l=1}^{N} a_l z^{-l}} = \frac{b_0 + b_1 z^{-1}}{(1 - a_1 z^{-1} - a_2 z^{-2})} \tag{58}$$

We will now try to get $H(z)$ into a form that will allow us to apply property (35) to write down $h[n]$ by inspection. We begin by factoring the denominator in (58) into:

$$(1 - a_1 z^{-1} - a_2 z^{-2}) = (1 - r_1 z^{-1})(1 - r_2 z^{-1}) \tag{59}$$

where r_1 and r_2 are the roots of the polynomial equation,

$$z^2(1 - a_1 z^{-1} - a_2 z^{-2}) = z^2 - a_1 z - a_2 = 0, \tag{60}$$

and are known as the *poles* of the system. Recall that for a polynomial of the form,

$$ax^2 + bx + c = 0 \tag{61}$$

the quadratic formula gives us the roots of equation (61) as,

$$r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \tag{62}$$

Applying the result in (62) to equation (60), where $a = 1$, $b = -a_1$ and $c = -a_2$, we get the following roots for equation (59):

$$r_1 = \frac{a_1 - \sqrt{a_1^2 + 4a_2}}{2} \text{ and } r_2 = \frac{a_1 + \sqrt{a_1^2 + 4a_2}}{2}. \tag{63}$$

Next, we want to express $H(z)$ as follows:

$$H(z) = \frac{b_0 + b_1 z^{-1}}{(1 - r_1 z^{-1})(1 - r_2 z^{-1})} = \frac{A_1}{(1 - r_1 z^{-1})} + \frac{A_2}{(1 - r_2 z^{-1})} \tag{64}$$

where we need to determine the values of A_1 and A_2. The expression on the right-hand side of equation (64) will allow us to express $h[n]$ as:

$$h[n] = A_1 r_1^n u[n] + A_2 r_2^n u[n] \text{ [property (35)].} \tag{65}$$

Below, we derive values for A_1 and A_2, starting from equation (64):

$$\frac{b_0 + b_1 z^{-1}}{(1 - r_1 z^{-1})(1 - r_2 z^{-1})} = \frac{A_1}{(1 - r_1 z^{-1})}\left(\frac{1 - r_2 z^{-1}}{1 - r_2 z^{-1}}\right) + \frac{A_2}{(1 - r_2 z^{-1})}\left(\frac{1 - r_1 z^{-1}}{1 - r_1 z^{-1}}\right) \tag{66}$$

$$\frac{b_0 + b_1 z^{-1}}{(1 - r_1 z^{-1})(1 - r_2 z^{-1})} = \frac{A_1(1 - r_2 z^{-1}) + A_2(1 - r_1 z^{-1})}{(1 - r_1 z^{-1})(1 - r_2 z^{-1})} \tag{67}$$

$$b_0 + b_1 z^{-1} = A_1(1 - r_2 z^{-1}) + A_2(1 - r_1 z^{-1}) \tag{68}$$

$$b_0 + b_1 z^{-1} = (A_1 + A_2) + (-A_1 r_2 - A_2 r_1) z^{-1} \tag{69}$$

Note that in order for equation (69) to hold, the following two equations must be true:

$$A_1 + A_2 = b_0 \tag{70}$$

$$-A_1 r_2 - A_2 r_1 = b_1 \tag{71}$$

Equations (70) and (71) are two linear equations in A_1 and A_2 and can be solved for A_1 and A_2:

$$A_1 = \frac{b_1 + b_0 r_1}{r_1 - r_2}, \; A_2 = \frac{b_1 + b_0 r_2}{r_2 - r_1} \tag{72}$$

Therefore, the impulse response $h[n]$ for IIR system (57) is given by,

$$h[n] = A_1 r_1^n u[n] + A_2 r_2^n u[n] \tag{73}$$

where,

$$r_1 = \frac{a_1 - \sqrt{a_1^2 + 4a_2}}{2}, \; r_2 = \frac{a_1 + \sqrt{a_1^2 + 4a_2}}{2}, \; A_1 = \frac{b_1 + b_0 r_1}{r_1 - r_2} \text{ and } A_2 = \frac{b_1 + b_0 r_2}{r_2 - r_1}. \tag{74}$$

Note that the solution for $h[n]$ in equations (73) and (74) only works for distinct roots, i.e. $r_1 \neq r_2$; also note that equations (73) and (74) completely specify the impulse response of the system in terms of the filter coefficients a_1, a_2, b_0 and b_1.

In the following section, we consider some numeric examples for this second-order example.

E. Numeric examples (second-order system)

We now consider four different numeric examples, for which we have already computed the intermediate values, as shown in the table below.

	a_1	a_2	b_0	b_1	r_1	r_2	A_1	A_2
#1	0	9/16	2	1	-3/4	3/4	1/3	5/3
#2	4/5	-16/25	2	1	$\frac{4}{5}e^{-\mathbf{j}(\pi/3)}$	$\frac{4}{5}e^{\mathbf{j}(\pi/3)}$	$1 + \frac{\mathbf{j}3\sqrt{3}}{4}$	$1 - \frac{\mathbf{j}3\sqrt{3}}{4}$
#3	3/2	-1/2	2	1	1/2	1	-4	6
#4	8/5	-11/20	2	1	1/2	11/10	-10/3	16/3

For each of these numeric examples, the impulse response $h[n]$ is given by equation (73) above. Note that for example #2, we can express $h[n]$ in terms of a real-valued, discrete-time cosine function:

$$h[n] = A_1 r_1^n u[n] + A_2 r_2^n u[n] = \left[\left(1 + \frac{\mathbf{j}3\sqrt{3}}{4}\right)\left(\frac{4}{5}e^{-\mathbf{j}(\pi/3)}\right)^n + \left(1 - \frac{\mathbf{j}3\sqrt{3}}{4}\right)\left(\frac{4}{5}e^{\mathbf{j}(\pi/3)}\right)^n\right]u[n] \tag{75}$$

Note that,

$$1 + \frac{\mathbf{j}3\sqrt{3}}{4} = \frac{\sqrt{43}}{4}e^{\mathbf{j}\,\mathrm{atan}(3\sqrt{3}/4)} \text{ and } 1 - \frac{\mathbf{j}3\sqrt{3}}{4} = \frac{\sqrt{43}}{4}e^{-\mathbf{j}\,\mathrm{atan}(3\sqrt{3}/4)} \tag{76}$$

so that,

$$h_2[n] = \left[\frac{\sqrt{43}}{4}e^{\mathbf{j}\,\mathrm{atan}(3\sqrt{3}/4)}\left(\frac{4}{5}e^{-\mathbf{j}(\pi/3)}\right)^n + \frac{\sqrt{43}}{4}e^{-\mathbf{j}\,\mathrm{atan}(3\sqrt{3}/4)}\left(\frac{4}{5}e^{\mathbf{j}(\pi/3)}\right)^n\right]u[n] \tag{77}$$

$$h_2[n] = \left[\left(\frac{\sqrt{43}}{4}\right)\left(\frac{4}{5}\right)^n e^{-\mathbf{j}(\pi n/3 - \mathrm{atan}(3\sqrt{3}/4))} + \left(\frac{\sqrt{43}}{4}\right)\left(\frac{4}{5}\right)^n e^{\mathbf{j}(\pi n/3 - \mathrm{atan}(3\sqrt{3}/4))}\right]u[n] \tag{78}$$

$$h_2[n] = \left(\frac{\sqrt{43}}{2}\right)\left(\frac{4}{5}\right)^n \cos[\pi n/3 - \mathrm{atan}(3\sqrt{3}/4)]u[n]. \tag{79}$$

The other impulse responses are given by,

$$h_1[n] = \frac{1}{3}\left(-\frac{3}{4}\right)^n u[n] + \frac{5}{3}\left(\frac{3}{4}\right)^n u[n] \tag{80}$$

$$h_3[n] = -4\left(\frac{1}{2}\right)^n u[n] + 6(1)^n u[n] = \left[6 - 4\left(\frac{1}{2}\right)^n\right]u[n] \tag{81}$$

$$h_4[n] = -\frac{10}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{16}{3}\left(\frac{11}{10}\right)^n u[n]. \tag{82}$$

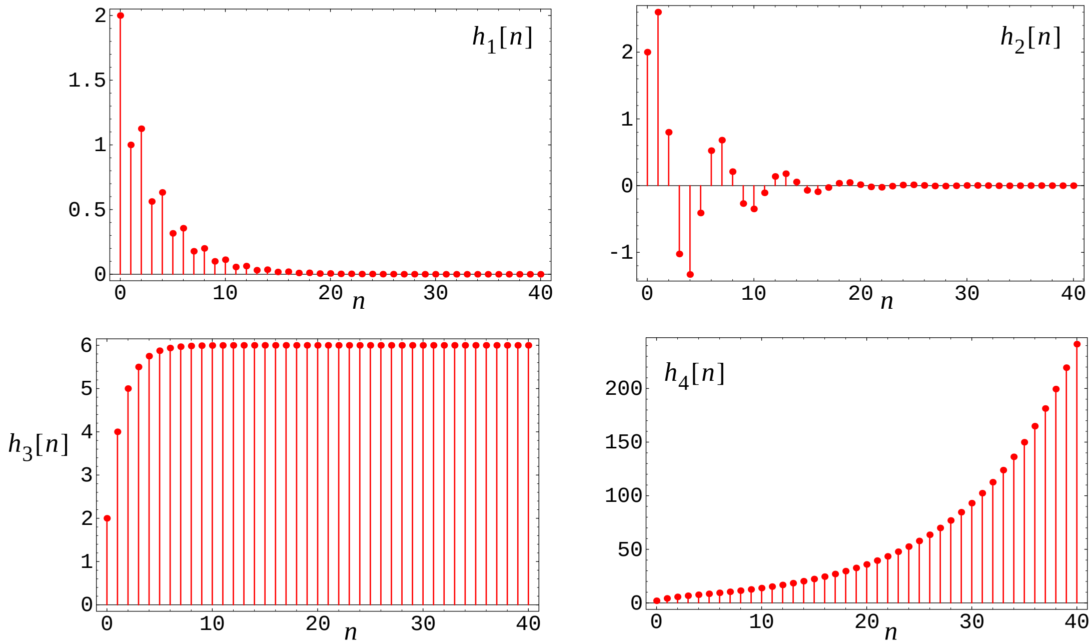

Figure 1

In Figure 1, we plot the impulse responses for each of the above cases, and make two observations. First, note that without the z-transform, we could have never obtained the analytic expressions for $h_i[n]$, $i \in \{1, 2, 3, 4\}$. Second, the roots r_1 and r_2 are intimately tied to the stability (or lack thereof) of each system. From (73), we see that when,

$$|r_i| < 1 \text{ (i.e. all the roots lie inside the unit circle of the complex plane)} \tag{83}$$

the system's impulse response will decay to zero as $n \to \infty$ (examples #1 and #2); while for,

$$|r_i| \geq 1 \text{ (i.e. the roots lie on or outside the unit circle of the complex plane)} \tag{84}$$

the system's impulse response will not decay to zero as $n \to \infty$ (examples #3 and #4), and may, in fact grow without bound (example #4). Therefore, by factoring the denominator and examining the resulting roots of $H(z)$, we can tell whether or not the system will be *BIBO (bounded-input, bounded-output) stable* — that is, whether or not the output signal is bounded,

$$|y[n]| < \infty, \ \forall n, \tag{85}$$

when the input signal and filter coefficients are bounded:

$$|x[n]| < \infty, \ \forall n, \tag{86}$$

$$|a_l| < \infty, \ \forall l, \tag{87}$$

$$|b_k| < \infty, \ \forall k. \tag{88}$$

In general, a system will be BIBO stable if all of the roots (or *poles*) of its transfer function $H(z)$ lie inside the unit circle of the complex plane; that is, if $|r_i| < 1$, $\forall i$.[1]

1. The case of $|r_i| = 1$ requires additional analysis and will not be considered separately here; whether or not such a system is BIBO stable depends on the multiplicity of the root on the unit circle.

F. Generalization to Nth-order systems

In the previous sections, we derived the time-domain impulse response $h[n]$ for a second-order IIR difference equation[1], by using the following procedure:

1. Compute $H(z)$ [equation (48)].
2. Factor the denominator polynomial of $H(z)$ and express the pole factors in the form:

$$(1 - r_1 z^{-1}) \text{ and } (1 - r_2 z^{-1}) . \tag{89}$$

3. Expand $H(z)$ into a partial fraction expansion of the form:

$$H(z) = \frac{A_1}{(1 - r_1 z^{-1})} + \frac{A_2}{(1 - r_2 z^{-1})} \tag{90}$$

4. Write down the time-domain impulse response $h[n]$:

$$h[n] = A_1 r_1^n u[n] + A_2 r_2^n u[n] . \tag{91}$$

Below, we generalize the above procedure to IIR systems of the general form,

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k] \tag{92}$$

with the restriction that $M < N$ and that the poles of the transfer function are distinct, i.e.,

$$r_i \neq r_j ,\ i, j \in \{1, 2, \ldots, N\} . \tag{93}$$

For such systems, we can compute $h[n]$ as follows:

1. Compute $H(z)$ [equation (48)].
2. Factor the denominator polynomial of $H(z)$ and express the pole factors in the form:

$$(1 - r_i z^{-1}) ,\ i \in \{1, 2, \ldots, N\} . \tag{94}$$

3. Expand $H(z)$ into a partial fraction expansion of the form:

$$H(z) = \sum_{i=1}^{N} \frac{A_i}{1 - r_i z^{-1}} , \text{ where } A_i = H(z)(1 - r_i z^{-1})\Big|_{z = r_i} \tag{95}$$

4. Write down the time-domain impulse response $h[n]$:

$$h[n] = \sum_{i=1}^{N} A_i r_i^n u[n] . \tag{96}$$

Note that the only difference between the procedure for the second-order system and the Nth order system is that we have introduced a general procedure for computing A_i in equation (95). Let us see why this works on the second-order example from before. From equation (64):

$$H(z) = \frac{b_0 + b_1 z^{-1}}{(1 - r_1 z^{-1})(1 - r_2 z^{-1})} = \frac{A_1}{(1 - r_1 z^{-1})} + \frac{A_2}{(1 - r_2 z^{-1})} . \tag{97}$$

Following equation (95), let us multiply both sides of equation (97) by $(1 - r_1 z^{-1})$ (in order to compute A_1):

---

1. Assuming distinct poles such that $r_1 \neq r_2$.

$$(1-r_1z^{-1})\frac{b_0+b_1z^{-1}}{(1-r_1z^{-1})(1-r_2z^{-1})} = (1-r_1z^{-1})\frac{A_1}{(1-r_1z^{-1})} + (1-r_1z^{-1})\frac{A_2}{(1-r_2z^{-1})} \quad (98)$$

$$\frac{b_0+b_1z^{-1}}{(1-r_2z^{-1})} = A_1 + A_2\frac{(1-r_1z^{-1})}{(1-r_2z^{-1})} \quad (99)$$

Now, substitute $z = r_1$ into equation (99):

$$\frac{b_0+b_1r_1^{-1}}{(1-r_2r_1^{-1})} = A_1 + A_2\frac{(1-r_1r_1^{-1})}{(1-r_2r_1^{-1})} \quad (100)$$

$$\frac{b_0+b_1r_1^{-1}}{(1-r_2r_1^{-1})} = A_1 + A_2\frac{(1-1)}{(1-r_2r_1^{-1})} = A_1 \quad (101)$$

$$A_1 = \frac{b_0+b_1r_1^{-1}}{(1-r_2r_1^{-1})} = \frac{b_0r_1+b_1}{r_1-r_2} \quad (102)$$

Note that this result is identical to that previously computed in equation (72). Equations (100) and (101) make clear why the formula,

$$A_i = H(z)(1-r_iz^{-1})\Big|_{z = r_i} \quad (103)$$

works in general when the roots are distinct. All of the unknown coefficients A_j, $j \neq i$ disappear, and we are left with a simple expression for A_i.

G. Output of system for finite-length input

Suppose we apply an input $x[n]$ of finite duration to an IIR LTI system:

$$x[n] = \sum_{k=0}^{L} c_k\delta[n-k] \quad (104)$$

Once we have computed the impulse response $h[n]$ of the IIR system, the output $y[n]$ is easy to compute by linearity and time-invariance (LTI) properties:

$$y[n] = \sum_{k=0}^{L} c_k h[n-k]\,. \quad (105)$$

4. Frequency response of IIR systems

A. Introduction

For a *stable* IIR system, the frequency response $H(e^{\mathbf{j}\theta})$ of that system is given by,

$$H(e^{\mathbf{j}\theta}) = H(z)\Big|_{z = e^{\mathbf{j}\theta}} \quad (106)$$

$$H(e^{\mathbf{j}\theta}) = \frac{\sum_{k=0}^{M} b_k e^{-\mathbf{j}k\theta}}{1-\sum_{l=1}^{N} a_l e^{-\mathbf{j}l\theta}} \quad (107)$$

Note that equation (107) reduces to DTFT of $h[n]$ for FIR systems (i.e. $a_l = 0$, $\forall l$).