Lecture #6: Continuous-Time Signals

1. Introduction

In this lecture, we discussed the following topics:

1. Mathematical representation and transformations of continuous-time signals.

2. Some important continuous-time functions.

As we have seen so far, signals are usually one-dimensional functions of time.[1] We can broadly classify these signals into two categories:

1. *Continuous-time signals*, denoted as $x(t)$, where t usually denotes time as the independent variable, and

2. *Discrete-time signals*, denoted as $x[n]$, where n is an integer and denotes the time index.

Signals in the real world are usually continuous-time. Discrete-time signals, on the other hand, are convenient to represent and operate on in digital computer systems; oftentimes, discrete-time signals are sampled versions of continuous-time signals. Because of the computer revolution of the past several decades, the study and understanding of discrete-time signals and systems has gone up dramatically with the rise of ever-more powerful computers and DSP (digital signal processing) systems.

We have already seen (and, in some cases, heard) many examples of signals, both continuous-time and discrete-time. In today's lecture, we begin our study of signals from a more mathematical point of view.

2. Continuous-time signals

A. Signal transformations

One of the most important basic skills we require in our study of signals is the ability to understand basic transformations with respect to the independent variable t. In this section, we will examine transformations on continuous-time signals; later, we will do the same for discrete-time signals.

Let $x(t)$ denote a continuous function of time t. In this class it will frequently be important to know how the function x changes when we change its argument. The table below gives the qualitative effect of some simple changes in argument.

function	*effect*
$x(-t)$	Reflection
$x(at)$, $a > 1$	Compression
$x(t/a)$, $a > 1$	Stretching
$x(t-a)$, $a > 0$	Shift to the right along the horizontal axis
$x(t+a)$, $a > 0$	Shift to the left along the horizontal axis
$a \cdot x(t)$, $a > 1$	Magnification
$a \cdot x(t)$, $a < 1$	Reduction
$x(t) + a$, $a > 0$	Shift up along the vertical axis
$x(t) - a$, $a > 0$	Shift down along the vertical axis

1. *We have seen one exception — namely, images which are two-dimensional signals, where the independent variables are the horizontal and vertical location of each pixel* (x, y) *and the dependent variable is each pixel's color (intensity for gray-scale images).*

Figures 1 and 2 illustrate some of these on two simple continuous functions. It is very important that you understand each of these illustrations, and are able to perform them yourself without the aid of a computer or calculator.

Compound transformations that perform both scaling and left/right shifting are a little trickier than each one by itself. Consider $x(t)$ in Figure 1 and the compound transformation $x(2t-1)$. To understand what this function looks like, we first change it to:

$$x(2t-1) = x\left[2\left(t-\frac{1}{2}\right)\right] \tag{1}$$

In this form, we see that we first scale the function (in this case, compress it), and then shift the scaled function by $1/2$ time units (*not* 1 time units) to the right; this transformation is illustrated in the bottom right corner of Figure 1. Figure 2 (bottom right corner) illustrates another compound-transformation example:

$$x(-t+5) = x[-(t-5)] \tag{2}$$

Again, we see that by factoring the scaling information (in this case a reflection), the function is first reflected about the y-axis, and is then shifted 5 time units to the right (*not* to the left).

B. Some useful continuous-time signals

In this section, we introduce some very useful continuous-time signals. The first of these is the *impulse* or *Dirac delta* function $\delta(t)$. It is defined as follows:

$$\delta(t) = 0,\ t \neq 0 \tag{3}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{4}$$

Note that the value of $\delta(0)$ is defined only implicitly by equation (4). We can view the Dirac delta function as the limiting case of a square pulse $p(t)$, as illustrated in Figure 3 below.

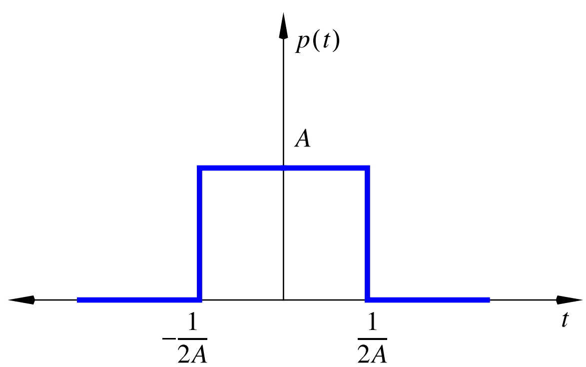

Figure 3

Note that no matter what the value of A, the integral of $p(t)$ will always be equal to one:

$$\int_{-\infty}^{\infty} p(t)dt = 1 \tag{5}$$

For larger values of A, $p(t)$ becomes more narrow and taller, but equation (5) still holds. Therefore, we can view $\delta(t)$ as the following limit:

$$\lim_{A \to \infty} p(t) = \delta(t) \tag{6}$$

One of the important properties of the $\delta(t)$ function is its *sifting* or *sampling property*. For any continuous-time function $x(t)$,

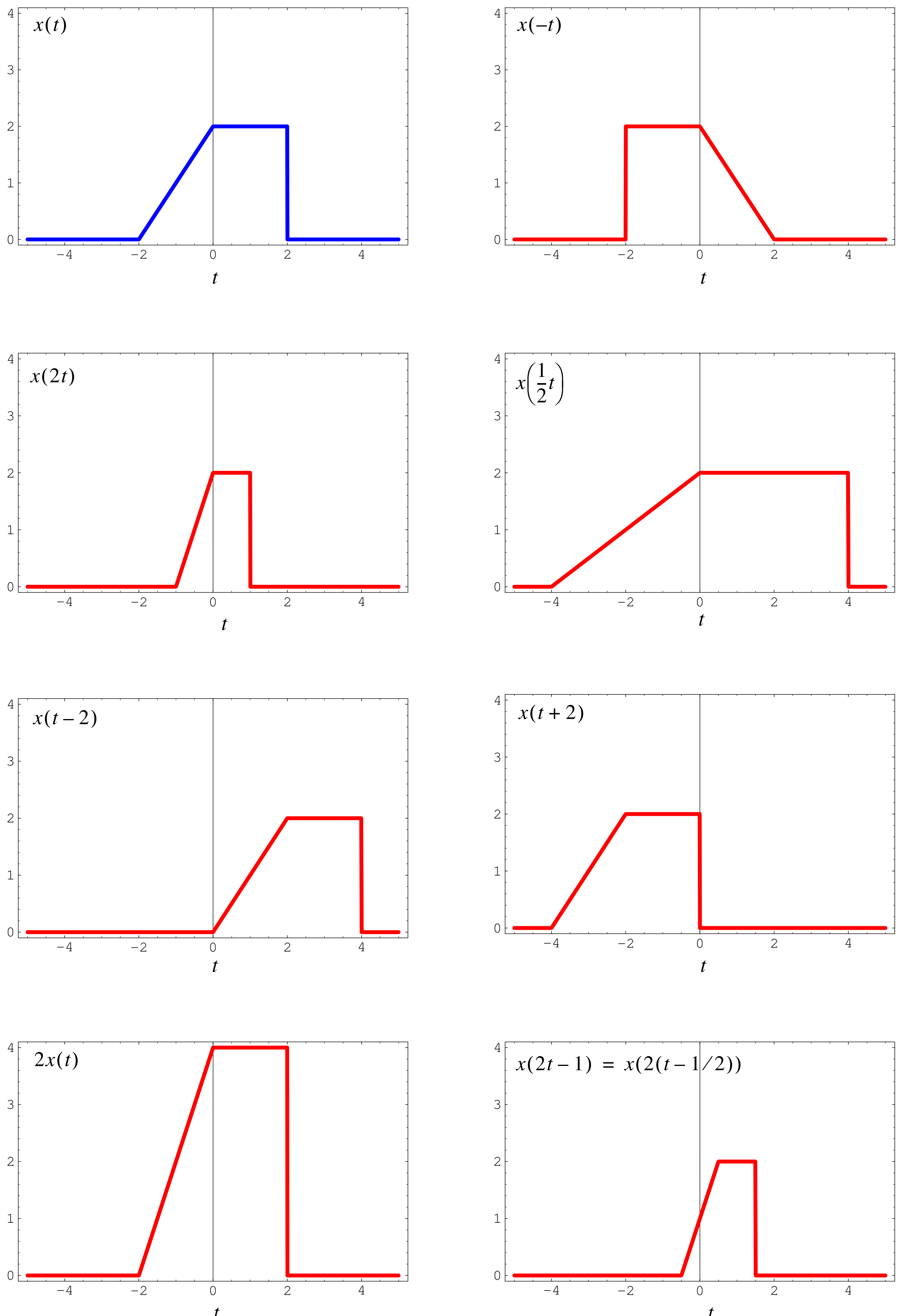

Figure 1

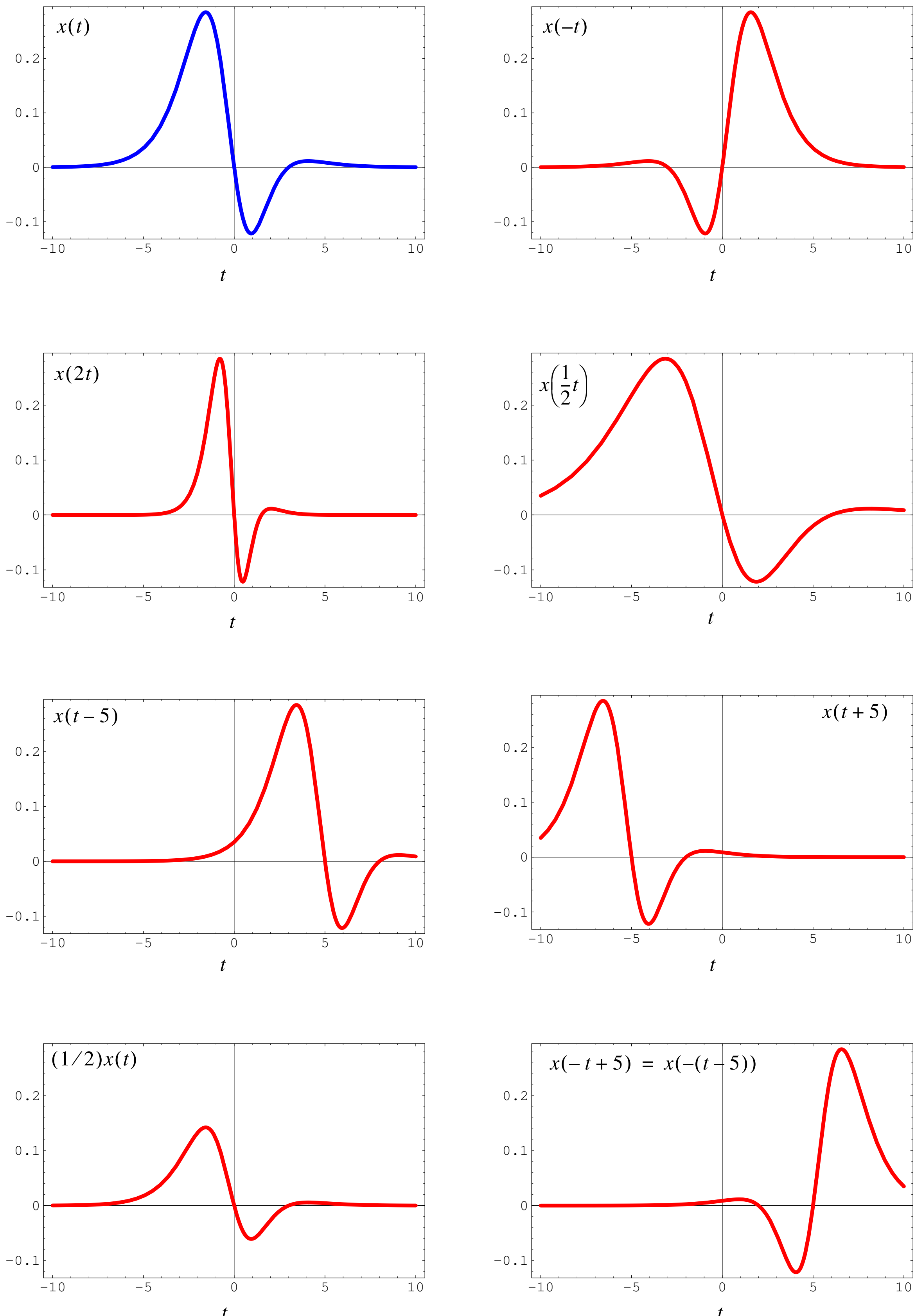

Figure 2

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0) \tag{7}$$

since the time-shifted delta function $\delta(t-t_0)$ is zero everywhere except for $t = t_0$. Let's look at a couple of examples:

$$\int_{-\infty}^{\infty} A\delta(t-t_0)dt = A \quad (x(t) = A = \text{some constant}) \tag{8}$$

$$\int_{-\infty}^{\infty} x(t)\delta(t-2)dt = x(2) \tag{9}$$

$$\int_{-\infty}^{\infty} x(t-2)\delta(t)dt = x(-2) \tag{10}$$

$$\int_{-\infty}^{\infty} x(t-3)\delta(t+2)dt = x(-5) \tag{11}$$

One place where the delta function comes up is in the continuous-time Fourier transform representation of a sinusoid. For the sinusoid,

$$x(t) = A\cos(2\pi f_o t), \tag{12}$$

the magnitude spectrum representation $|X(f)|$ is given by,

$$|X(f)| = \frac{1}{2}A[\delta(f+f_0) + \delta(f-f_0)] \tag{13}$$

as illustrated in Figure 4 below.

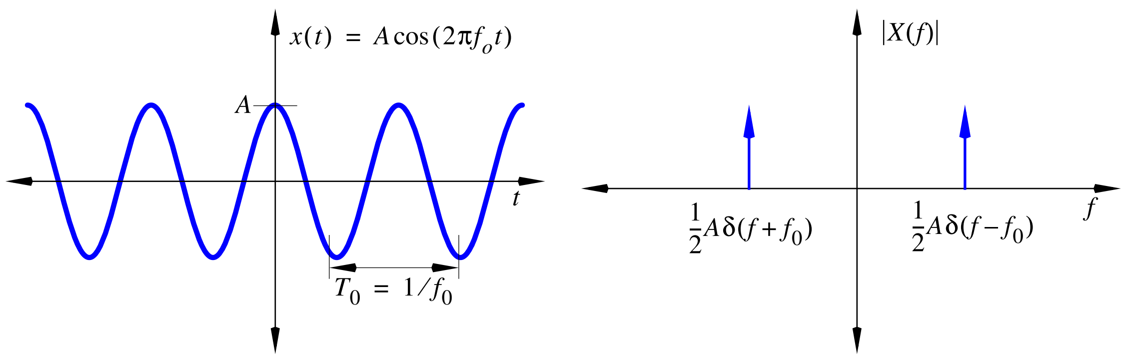

Figure 4

Another continuous-time function of significance in our mathematical representation of signals is the *unit step* function $u(t)$,

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{14}$$

plotted in Figure 5. The unit step function allows us to mathematically represent signals that are piece wise continuous with discontinuities at a finite number of points. Consider for example, the three functions plotted in Figure 6. Each of these would be impossible to describe functionally without the aid of the unit step function; with $u(t)$, however, each function has a straightforward representation:

$$x_1(t) = 4[u(t+3) - u(t-2)] \tag{15}$$

$$x_2(t) = (t+1)[u(t+1) - u(t)] + (-t+1)[u(t) - u(t-1)] \tag{16}$$

$$x_3(t) = \sin(2\pi t)[u(t) - u(t-1)] \tag{17}$$

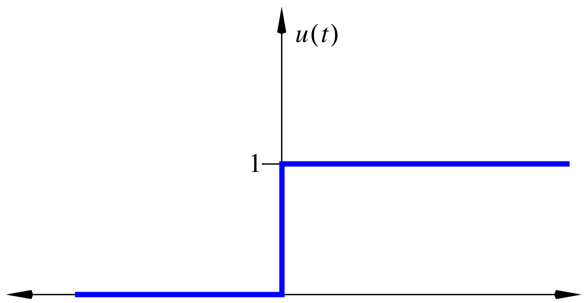

Figure 5: The unit-step function

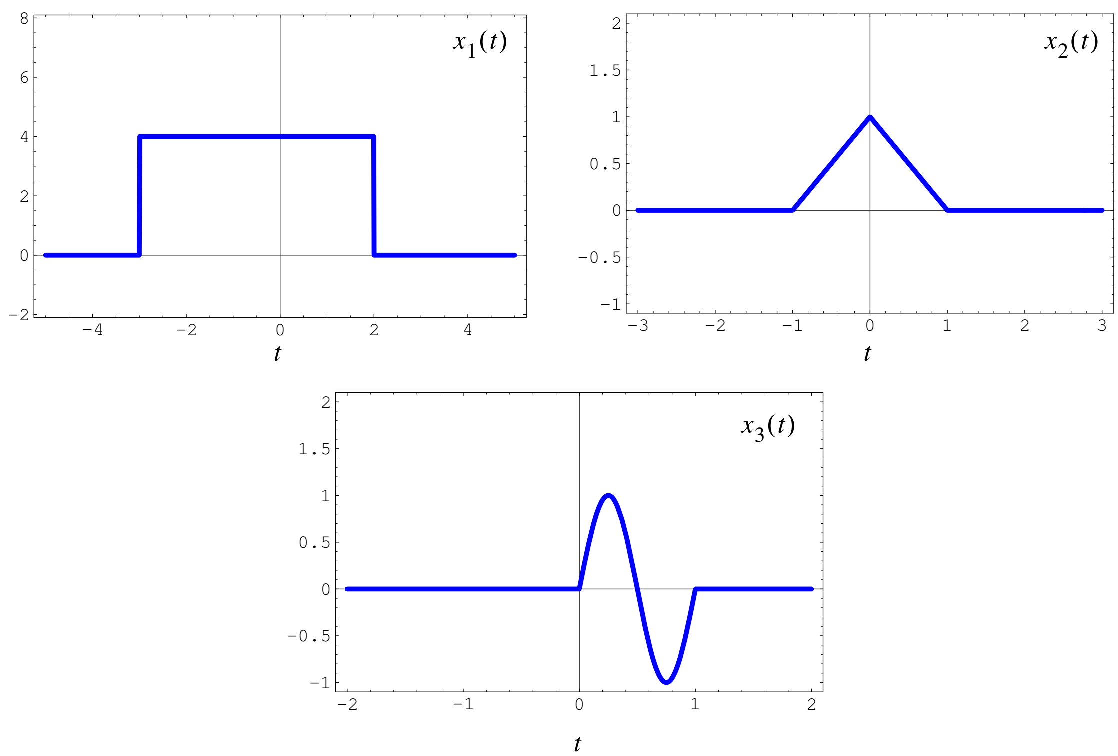

Figure 6

Finally, we look at sinusoidal functions and their mathematical representation. In our previous lectures, we have already seen and heard many different sinusoids and these functions are perhaps the most important in our understanding of signals and system analysis. While we could use both cosines or sines in our mathematical representation, we will primarily use the cosine function by convention from here on out. Consider $x(t)$ defined by,

$$x(t) = A\cos(2\pi f_o t + \alpha) \tag{18}$$

and plotted in Figure 7 below. For equation (18), we define the following quantities:

$$A = \textit{amplitude},\ \alpha = \textit{phase}\ \text{(rad)}, \tag{19}$$

$$f_o = \textit{cyclic frequency}\ \text{(1/sec or Hertz (Hz) units)}, \tag{20}$$

$$\omega_o = 2\pi f_o = \textit{radian frequency}\ \text{(rad/sec units)}, \tag{21}$$

$$T_o = 1/|f_o| = \textit{period}\ \text{(units of seconds)}. \tag{22}$$

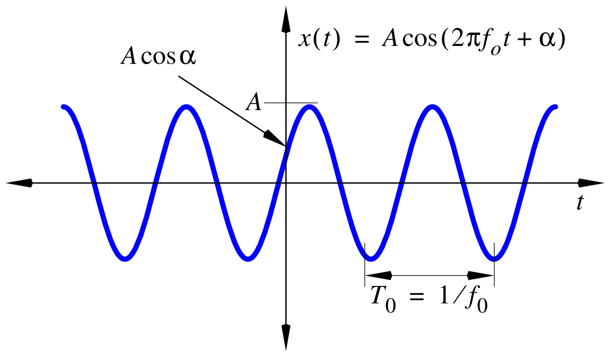

Figure 7

A sinusoidal wave is *periodic*; that is,

$$x(t) = x(t - T_o),\ \forall t. \tag{23}$$

In fact, the sinusoid is perhaps the most important periodic waveform that we will study, since almost all other periodic waveforms can be constructed as the infinite sum of sinusoids; for any periodic signal $x(t)$, we can write,

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(2\pi f_o k t) + b_k \sin(2\pi f_o k t) \tag{24}$$

Equation (24) is known as the *Fourier series* representation of the periodic signal $x(t)$. We will look at how to determine the coefficients a_k and b_k a little later.