Lecture #8: Sampling in the Frequency Domain

1. Introduction

In this lecture, we continued our discussion of sampling, specifically looking at the sampling process from a frequency-domain perspective.

2. Sampling in the frequency domain

Last time, we introduced the *Shannon Sampling Theorem* given below:

> ***Shannon Sampling Theorem****: A continuous-time signal $x_c(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x_c(nT_s)$, if the samples are taken at a sampling frequency $f_s > 2f_{max}$, that is, at a sampling frequency greater than $2f_{max}$. The frequency $2f_{max}$ is known as the Nyquist frequency.*

And, we demonstrated the sampling theorem visually by showing the reconstruction of a 1Hz cosine wave at various sampling frequencies above and below the Nyquist frequency.

In this lecture, we look at sampling in the frequency domain, to explain why we must sample a signal at a frequency greater than the Nyquist frequency. First, a basic result from sampling theory: When we sample a continuous-time signal $x_c(t)$, we can view the resulting *discrete-time* sequence $x[n]$ as a sequence of weighted and time-shifted impulses $x_s(t)$ in the *continuous-time* domain:

$$x_s(t) = \sum_n x[n]\delta\left(t - \frac{n}{f_s}\right) \tag{1}$$

The frequency spectrum $X_s(f)$ of $x_s(t)$ is given by,

$$X_s(f) = \sum_{k=-\infty}^{\infty} X_c(f - kf_s) \tag{2}$$

where $X_c(f)$ is the frequency spectrum of the continuous-time signal $x_c(t)$.[1] In other words, the frequency spectrum $X_c(f)$ of the original continuous-time signal is shifted by integer multiples of the sampling frequency f_s in the frequency spectrum $X_s(f)$.

To illustrate how this affects reconstruction of the original signal, let us once again consider a 1Hz cosine signal,

$$x_c(t) = \cos(2\pi t) \tag{3}$$

whose continuous frequency spectrum is given by,

$$X_c(f) = \frac{1}{2}\delta(f+1) + \frac{1}{2}\delta(f-1) \tag{4}$$

where $\delta(t)$ is the Dirac delta function. Both equations (3) and (4) are plotted in Figure 1 below.

In Figures 2 and 3 we illustrate sampling in the frequency domain for two sampling frequencies:

$$f_s = 10\text{Hz} > 2f_{max} \text{ (Figure 2)} \tag{5}$$

$$f_s = 1.6\text{Hz} < 2f_{max} \text{ (Figure 3)} \tag{6}$$

Let us consider the case of sufficient sampling first (e.g. $f_s = 10\text{Hz}$). In Figure 2(a), we plot the continuous-time representation $x_s(t)$ of the discrete-time sequence $x[n]$ for $f_s = 10\text{Hz}$, while Figure 2(b) plots the corresponding frequency spectrum $X_s(f)$ according to equation (2), where the original continuous-time spectrum $X_c(f)$ is indicated in blue. Now, observe from Figure 2(c) ($f_s = 10\text{Hz}$) that we can recover the original signal $x_c(t)$ from $x_s(t)$ by applying an ideal low-pass filter to $x_s(t)$ with cut-off frequencies $\pm f_s/2$; the consequence of this filter-

1. *Since proving equation (2) requires knowledge of the continuous-time Fourier transform, which is not a major topic of this course, we will only state this result without proof.*

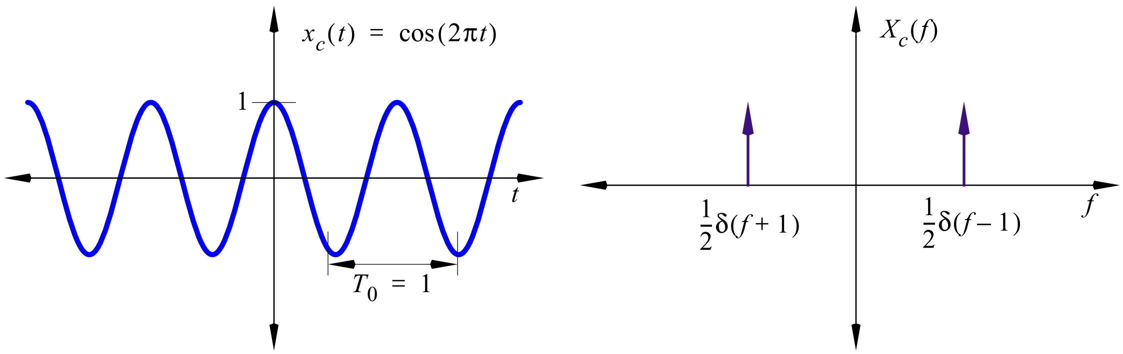

Figure 1

ing operation is shown in Figure 2(d), where $X_r(f)$ denotes the frequency spectrum of the reconstructed signal $x_r(t)$ plotted in Figure 2(e). Note that it can be shown[1] that the ideal low-pass filtering discussed above results in the following reconstructed signal $x_r(t)$:

$$x_r(t) = \sum_n x[n] \operatorname{sinc}[\pi f_s(t - n/f_s)] \tag{7}$$

Now, let us consider the case of insufficient sampling (e.g. $f_s = 1.6\text{Hz}$). Note from Figure 3 that the replicated spectral components in $X_s(f)$ now overlap the original spectrum $X_c(t)$ and that a low-pass filter with cut-off frequencies at $\pm f_s/2$ will no longer recover the original continuous-time signal, but one of its *aliases* at 0.6Hz, such that:

$$x_r(t) = \cos(2\pi \cdot 0.6t) \tag{8}$$

as plotted in Figure 3(e).

From these two illustrations, it should now be apparent why we must sample at a rate greater than the Nyquist frequency. When the sampling frequency is larger than the Nyquist frequency, the shifted spectral components in $X_s(f)$ that result from the sampling operation will not overlap the original spectrum, so that a low-pass filter can preserve the original spectrum $X_c(t)$. On the other hand, when the sampling frequency is less than the Nyquist frequency, the shifted spectral components will overlap the original spectrum and all the spectral components with frequencies above $f_s/2$ will be lost, while the recovered signal will contain *aliases* of the original signal. Note from Figure 3, that both the original signal $x_c(t)$ and the reconstructed signal $x_r(t)$ pass through all the sample points; in other words, $x_r(t)$ aliases $x_c(t)$.

3. Listening to aliasing

From Figure 3, we see that when we sample a continuous-time signal at a sampling frequency $f_s < 2f_{max}$, frequency components greater than f_{max} are folded into the frequency range $f \in [-f_s/2, f_s/2]$ as lower-frequency aliases of those higher frequencies. In music, for example, those aliases would be audible as distortions. Therefore, when digitizing music, it is important to limit the frequencies in that music selection *prior to* sampling.

In Figures 4 and 5 we illustrate a high-level block diagram of a CD recording and playback system. In recording music to a CD, we first low-pass filter the music with $f_{max} \approx 20\text{kHz}$, limiting the frequencies in the music to the audible range of the human ear. This step prevents the folding of higher frequency components, that we cannot hear, into the audible frequency range after sampling and reconstruction. Next, we sample the music at 44.1kHz , and record the sampled music as a sequence of quantized numbers, with $2^{14} = 16384$ levels of quantization; that is, we don't store the sampled values as floating-point reals, but rather as a sequence of integers in the range $[0, 16384]$. In playing back a CD, we convert the digitized music to the continuous-time domain with a low-pass filter that has cut-off frequencies at $\pm f_s/2$.

1. *To show this, we need to develop additional material; therefore we will defer mathematical derivation of this result to later in this course.*

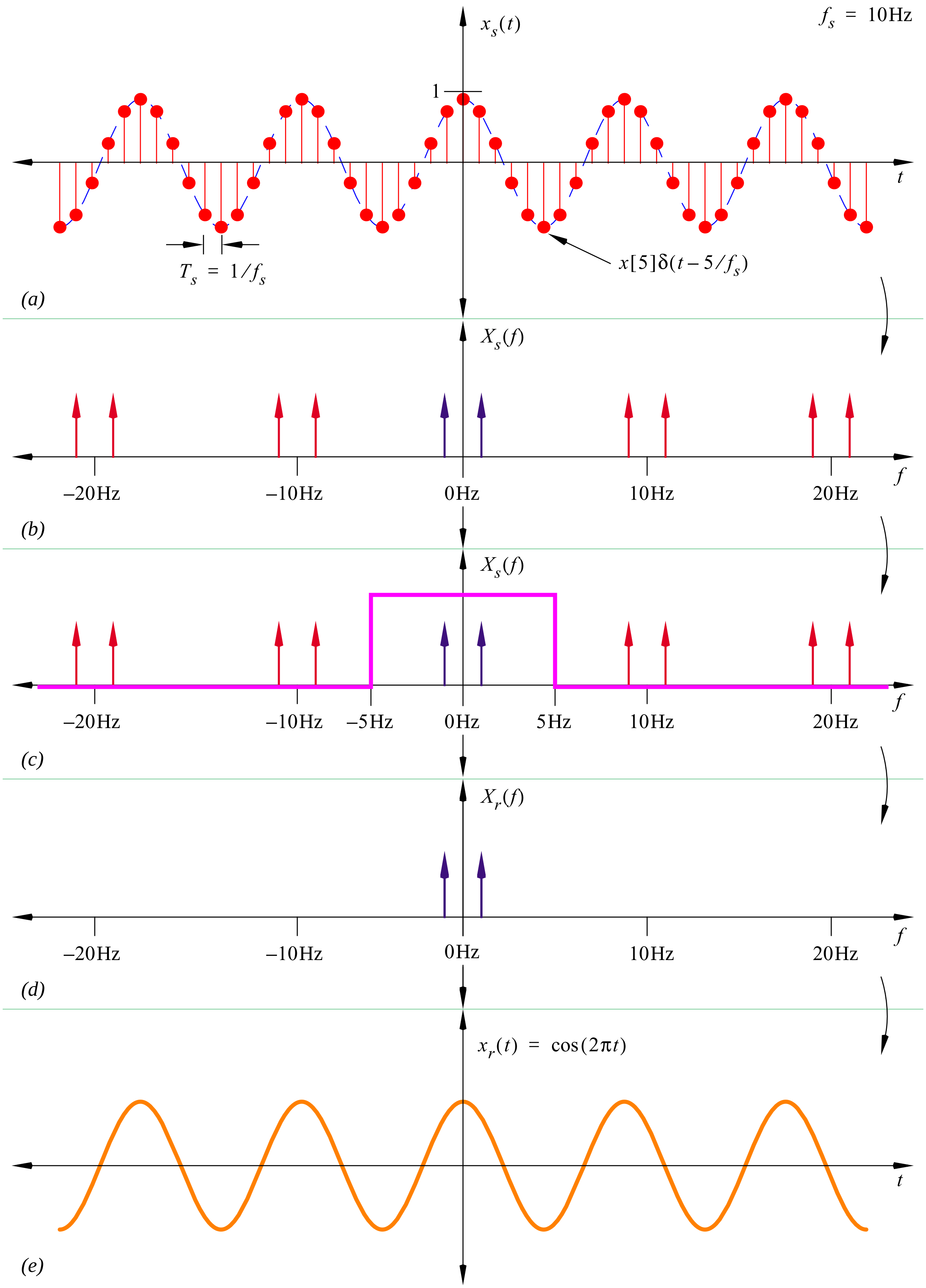

Figure 2: Sufficient sampling

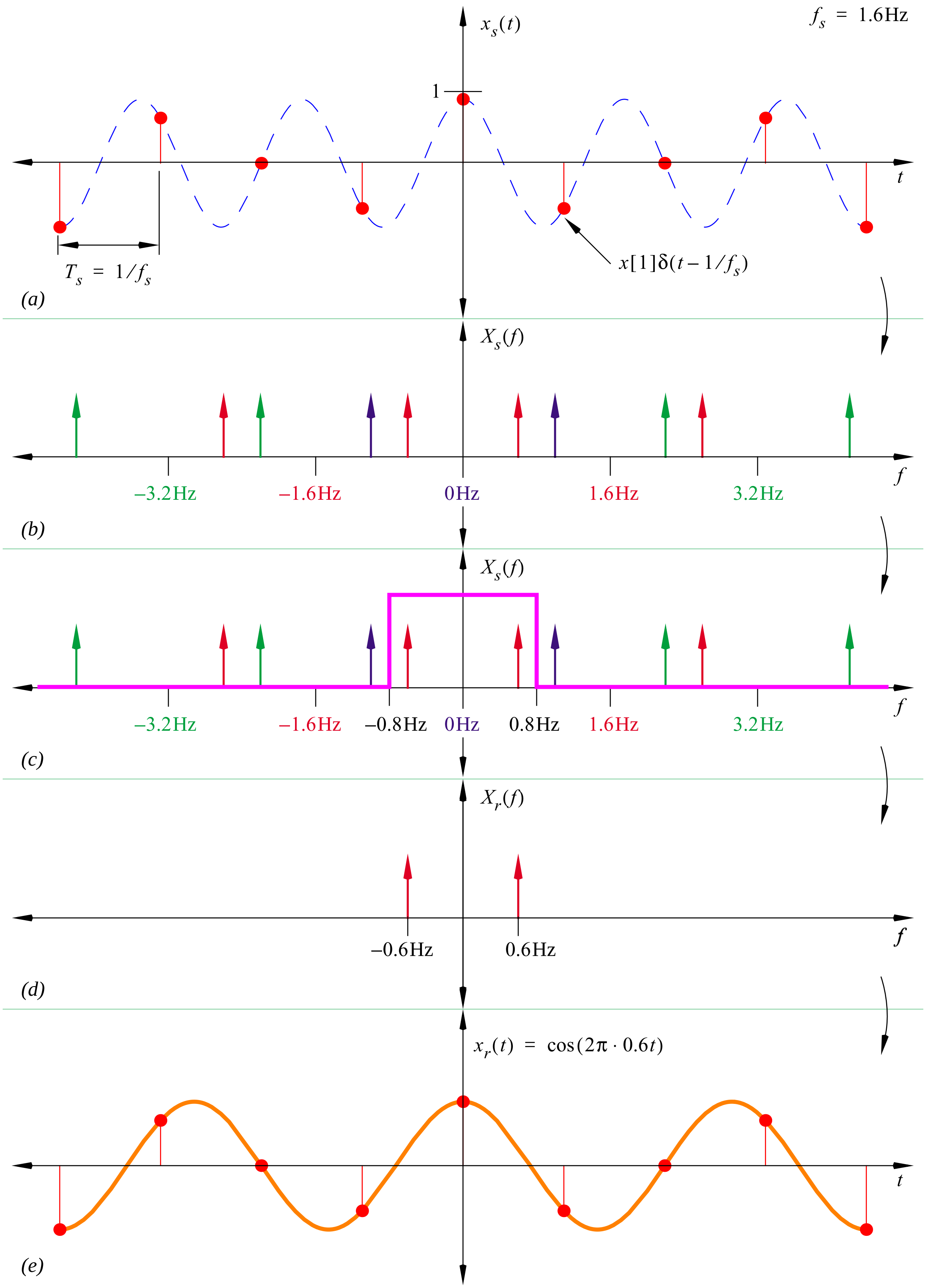

Figure 3: Insufficient sampling

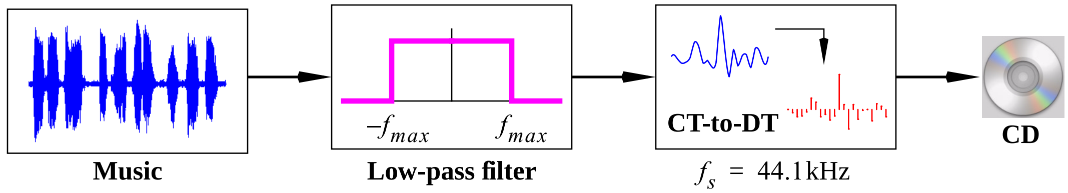

Figure 4: CD recording

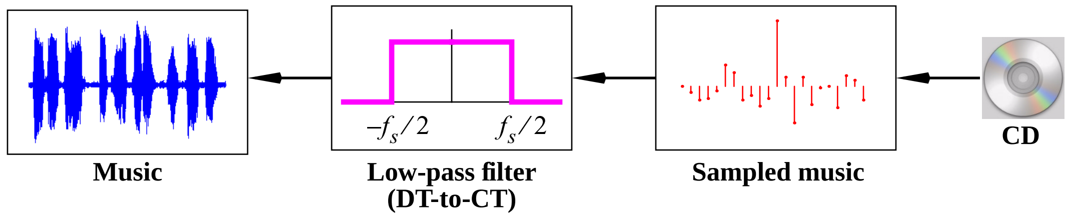

Figure 5: CD playback

If we did not low-pass filter the incoming music signal (as shown in Figure 4) prior to sampling, we should expect to hear distortions in the reconstructed music signal. In class, we demonstrated this on two short pieces of music: (1) a short segment of Kenny Roger's "The Gambler," and (2) a short segment of some Cantonese opera. We first resampled each piece of music to 48kHz (from 44.1kHz); note that this step does not add any information to the signal, but merely simplifies the process of down-sampling. Then, we down-sampled each piece of music to 8kHz, 4kHz and 2kHz in two different ways: (1) with low-pass filtering (as in Figure 4), and (2) without low-pass filtering. In class, we referred to the examples *with* low-pass filtering as *anti-aliased*, since the low-pass filtering prior to down-sampling ensures that no high-frequency aliases appear in the reconstructed signal. Listening to these two different versions at each lower sampling frequency, we could definitely hear distortion (aliasing) in the examples where no low-pass filtering was applied prior to down-sampling, while such distortion was not evident in the anti-aliased (low-pass filtered) examples. The original, as well as the down-sampled examples are all posted on the web site in *wav* and *mp3* formats.

4. Conclusion

Next time, we will conclude our discussion of sampling with some more examples, and then continue our mathematical treatment of discrete-time signals.