Lecture #9(a): Further Discussion on Sampling

1. Introduction

In this lecture, we discussed the following topics:

1. Additional examples of sampling in the frequency domain.
2. Further thoughts on the sampling process.
3. Some important discrete-time functions.
4. Introduction to complex variables.

This set of notes covers the first two topics, while other notes cover the remaining topics.

2. Additional examples of sampling in the frequency domain

Here we look at four more sampling case studies as listed below:

$$x_c(t) = \cos(2\pi t),\ f_s = 5\text{Hz} \text{ (Figure 1, sufficient sampling)} \tag{1}$$

$$x_c(t) = \cos(2\pi t),\ f_s = 1.5\text{Hz} \text{ (Figure 2, insufficient sampling)} \tag{2}$$

$$x_c(t) = \frac{\sin(\pi t)[2\pi t\cos(\pi t) + \sin(\pi t)]}{(\pi t)^2},\ f_s = 5\text{Hz} \text{ (Figure 3, sufficient sampling)} \tag{3}$$

$$x_c(t) = \frac{\sin(\pi t)[2\pi t\cos(\pi t) + \sin(\pi t)]}{(\pi t)^2},\ f_s = 1.5\text{Hz} \text{ (Figure 4, insufficient sampling)} \tag{4}$$

For each of these examples, we plot (1) the original signal $x_c(t)$, (2) its frequency spectrum $X_c(f)$, (3) the sampled frequency spectrum $X_s(f)$, (4) ideal low-pass filter with cut-off frequencies $\pm f_s/2$, (5) filtered frequency spectrum $X_r(f)$, and (6) the reconstructed signal $x_r(t)$ (in orange). The examples corresponding to Figures 1 and 2 are similar to the detailed examples presented in the previous lecture; the examples corresponding to Figures 3 and 4, however, are a little different, in that the frequency spectrum of $x_c(t)$ does not simply have nonzero components at two discrete frequencies, but rather for the entire frequency range from 0Hz to 1Hz. Note from Figure 4 that when we do not sample at a rate higher than the Nyquist frequency of 2Hz — namely 1.5Hz — frequency components higher than,

$$f_s/2 = 1.5/2 = 0.75\text{Hz} \tag{5}$$

in the original signal are folded into the frequency range $[-0.75\text{Hz}, 0.75\text{Hz}]$. Therefore, not only do we loose frequencies above $f_s/2 = 0.75\text{Hz}$, but frequency magnitudes in the range $[0.5\text{Hz}, 0.75\text{Hz}]$ are no longer representative of the original signal. (Figures 1 through 4 were generated with the *Mathematica* notebook “sampling_examples.nb”.)

3. Some further thoughts on sampling

A. Sampled-signal reconstruction: another look

We have previously said that the low-pass filtering step in the reconstruction of a sampled signal from the discrete-time to the continuous-time domain is equivalent to the following idealized reconstruction of the original signal:

$$x_r(t) = \sum_n x[n]\ \text{sinc}[\pi f_s(t - n/f_s)] \tag{6}$$

Below, we explore the reconstruction of a sampled signal with two non-ideal functions, a square pulse and triangle interpolation function, such that the reconstructed signals are given by,

$$x_r(t) = \sum_n x[n]\text{pulse}[f_s(t - n/f_s)] \tag{7}$$

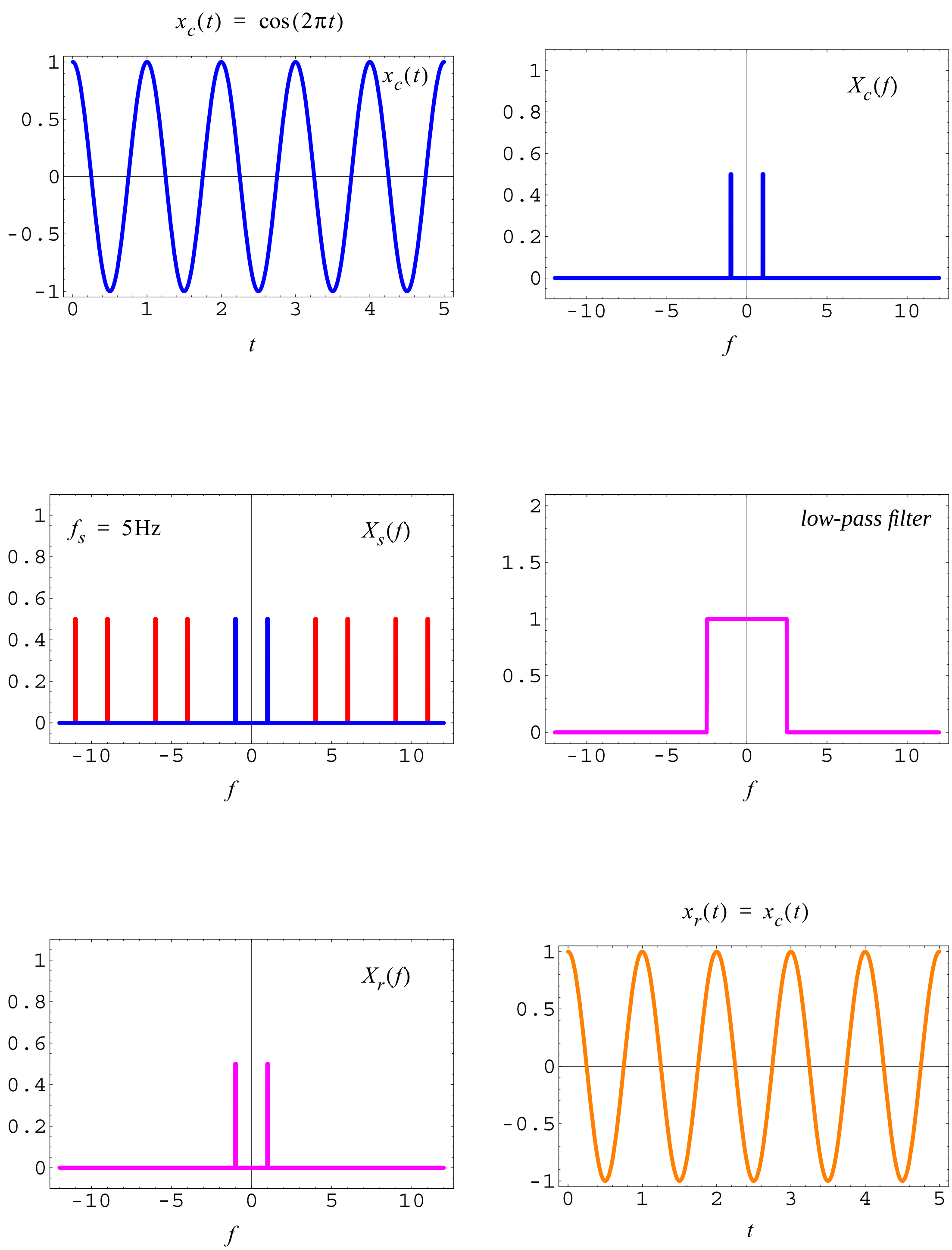


Figure 1: Sufficient sampling

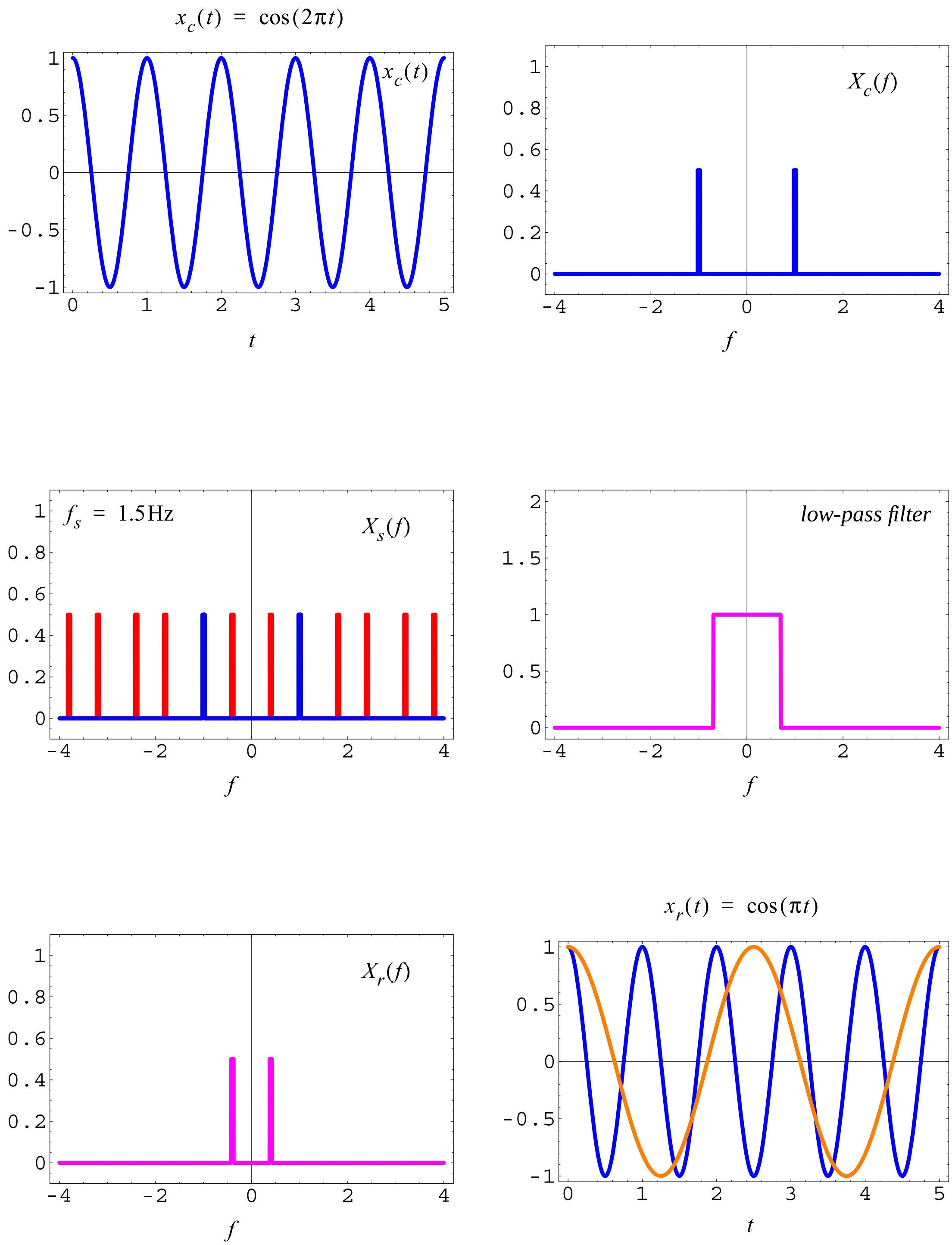


Figure 2: Insufficient sampling

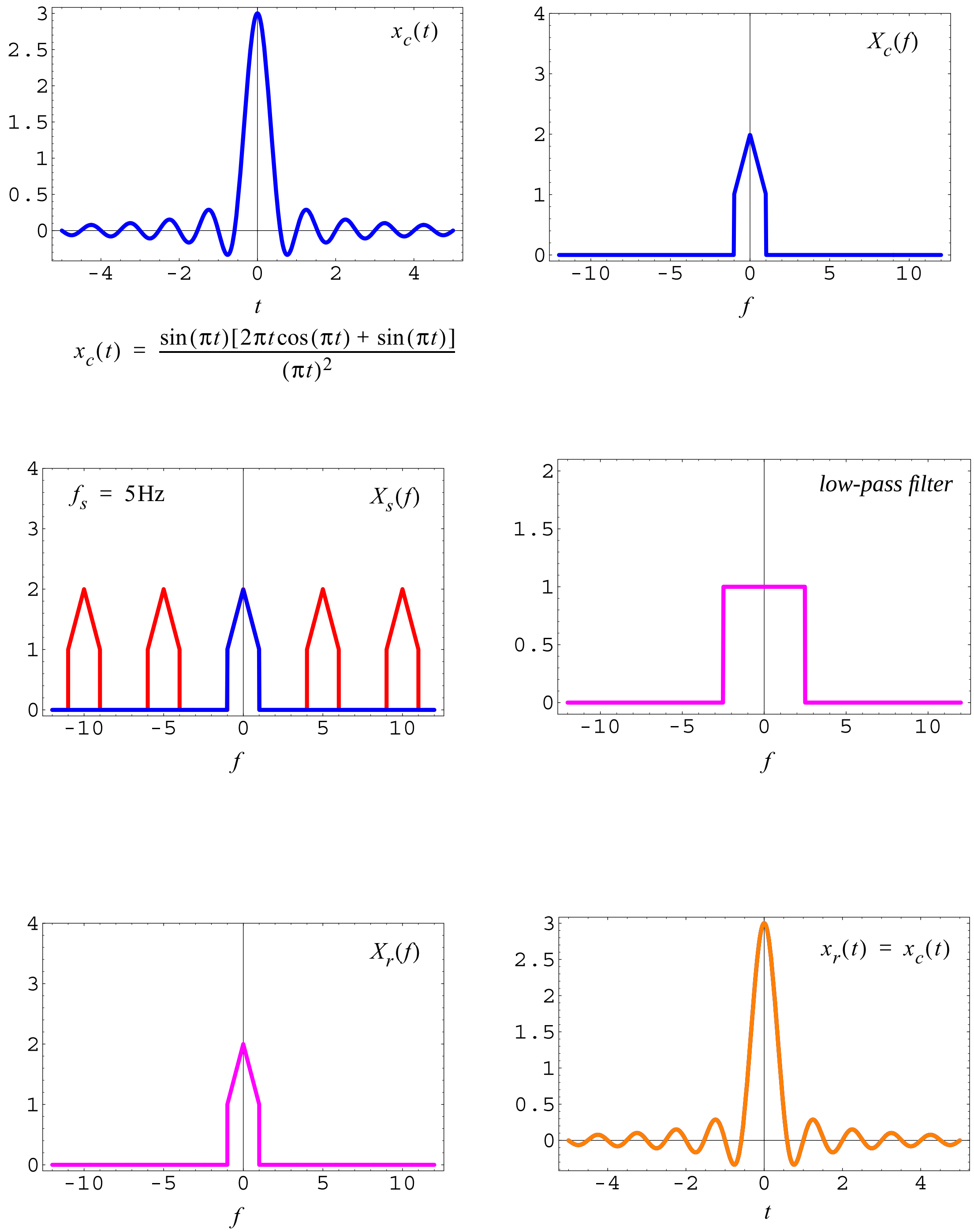


Figure 3: Sufficient sampling

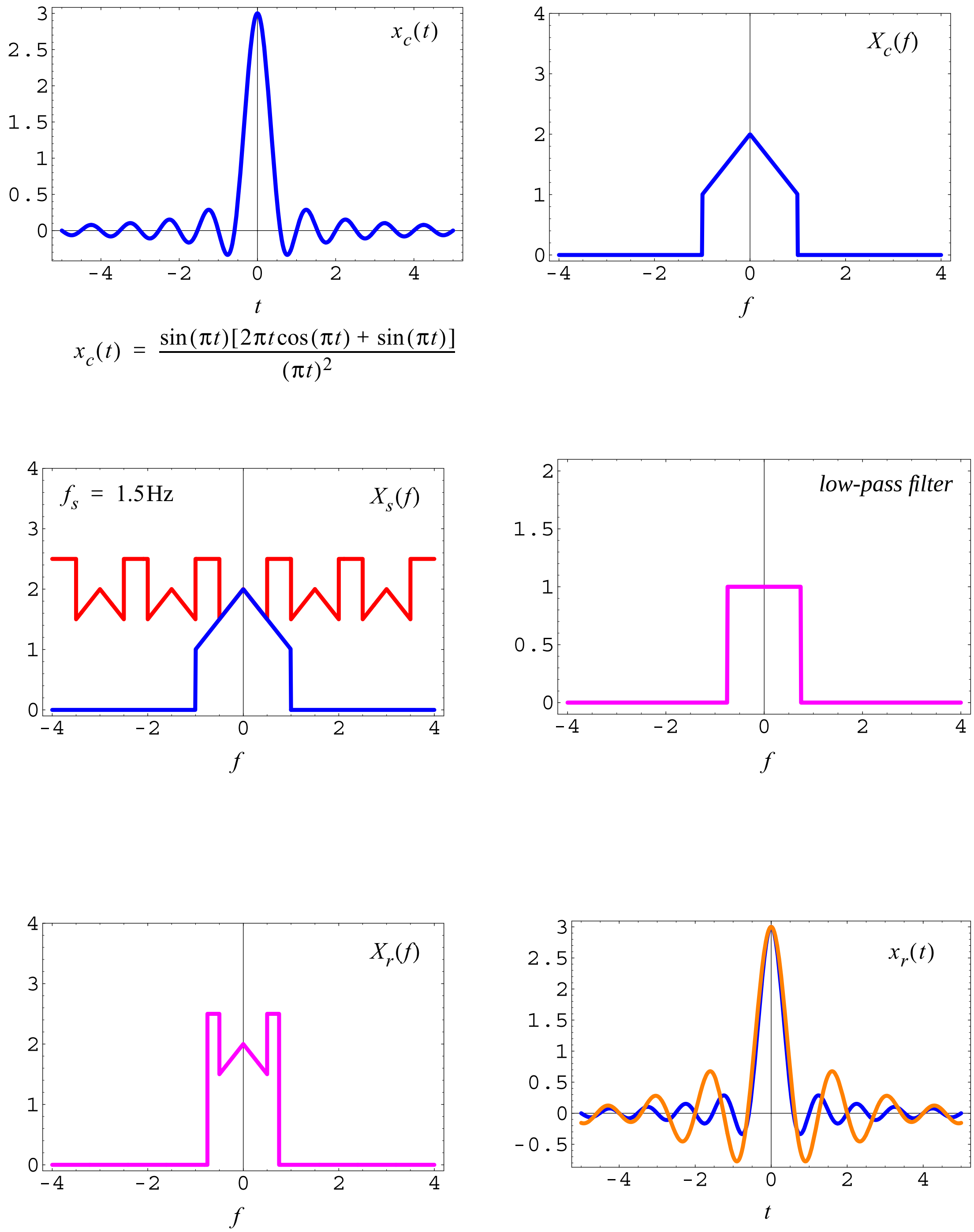


Figure 4: Insufficient sampling

and,

$$x_r(t) = \sum_n x[n]\text{triangle}[f_s(t - n/f_s)] \quad (8)$$

respectively. These reconstructions are different from equation (6) in that they have finite-length (in time), and therefore are more readily realizable in real systems. In Figures 5, 6 and 7, we illustrate reconstructions (6) through (8) for the continuous-time function,

$$x_c(t) = \cos(2\pi t) \quad (9)$$

and three different sampling frequencies $f_s = 2.3\text{Hz}, 6\text{Hz}, 10\text{Hz}$. Note that even though all three sampling frequencies are above the Nyquist sampling frequency for $x_c(t)$ in equation (9), the non-ideal reconstructions are relatively poor approximations of the original function for the lowest sampling frequency in Figure 5, although reconstruction (8) is better than reconstruction (7). Thus, in practice, typically two things are done to achieve high-fidelity signal reconstruction. First, an approximation of finite length (over time) of the $\text{sinc}(t)$ function is used as the interpolation function; this corresponds to non-ideal low-pass filtering. Second, we typically oversample (that is, sample at a frequency significantly higher than $2f_{max}$), since we can see from Figures 5 through 7 that the more samples we have, the better the reconstruction will be, even for non-ideal interpolation functions. (Figures 5 through 7 were generated with the *Mathematica* notebook "reconstruction.nb".)

B. Listening to subsampled music

During lecture, I attempted to illustrate, through music (a snippet of Chinese opera), that when we sample a signal at f_s, the only frequencies present in the reconstructed signal lie in the range $[-f_s/2, f_s/2]$. I took a piece of music from a CD (sampled at 44.1kHz) and subsampled it at 22.05kHz, 11.025kHz, 8kHz, 4kHz and 2kHz. According to sampling theory, the highest frequency components that can be reconstructed for each sound file are half of each sampling frequency. For example, for the music snippet with sampling rate $f_s = 8\text{kHz}$, we should not be able to hear any frequencies greater than 4kHz; we saw that this was indeed the case, by playing each subsampled piece of music in an mp3 player, and manipulating different frequency bands in the equalizer of the mp3 player. As an example, an 8kHz sampled signal should sound exactly the same for the two equalizer configurations pictured in Figure 8, and, indeed, it did.

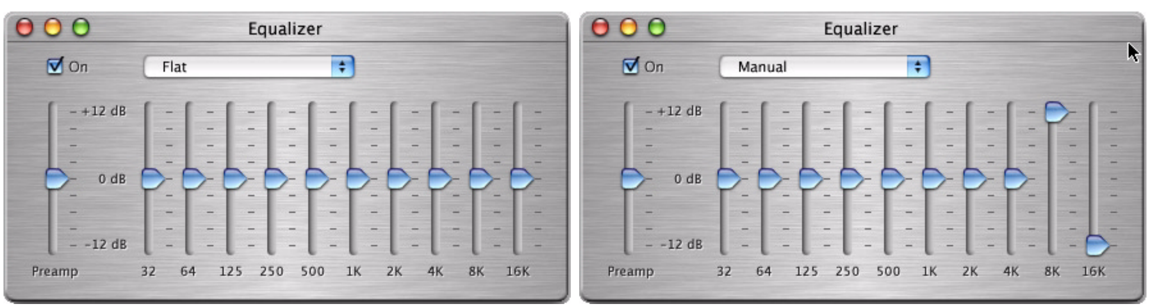


Figure 8

A perceptible difference in music sampled at lower frequencies is that it sounds muffled, or completely unrecognizable (as is the case for $f_s = 2\text{kHz}$). To listen for yourself, go to the web site, where all of the above referenced music files are posted in *wav* and *mp3* formats.

4. Concluding thoughts

The Sampling Theorem is one of the big ideas in this course, and it is important that you understand all the examples presented throughout the past few lectures. In summary, for a continuous-time signal, we must sample that signal at a sampling frequency $f_s > 2f_{max}$. Conversely, a signal that has been reconstructed from a sampled signal with sampling frequency f_s will not contain any frequencies higher than $f_s/2$.

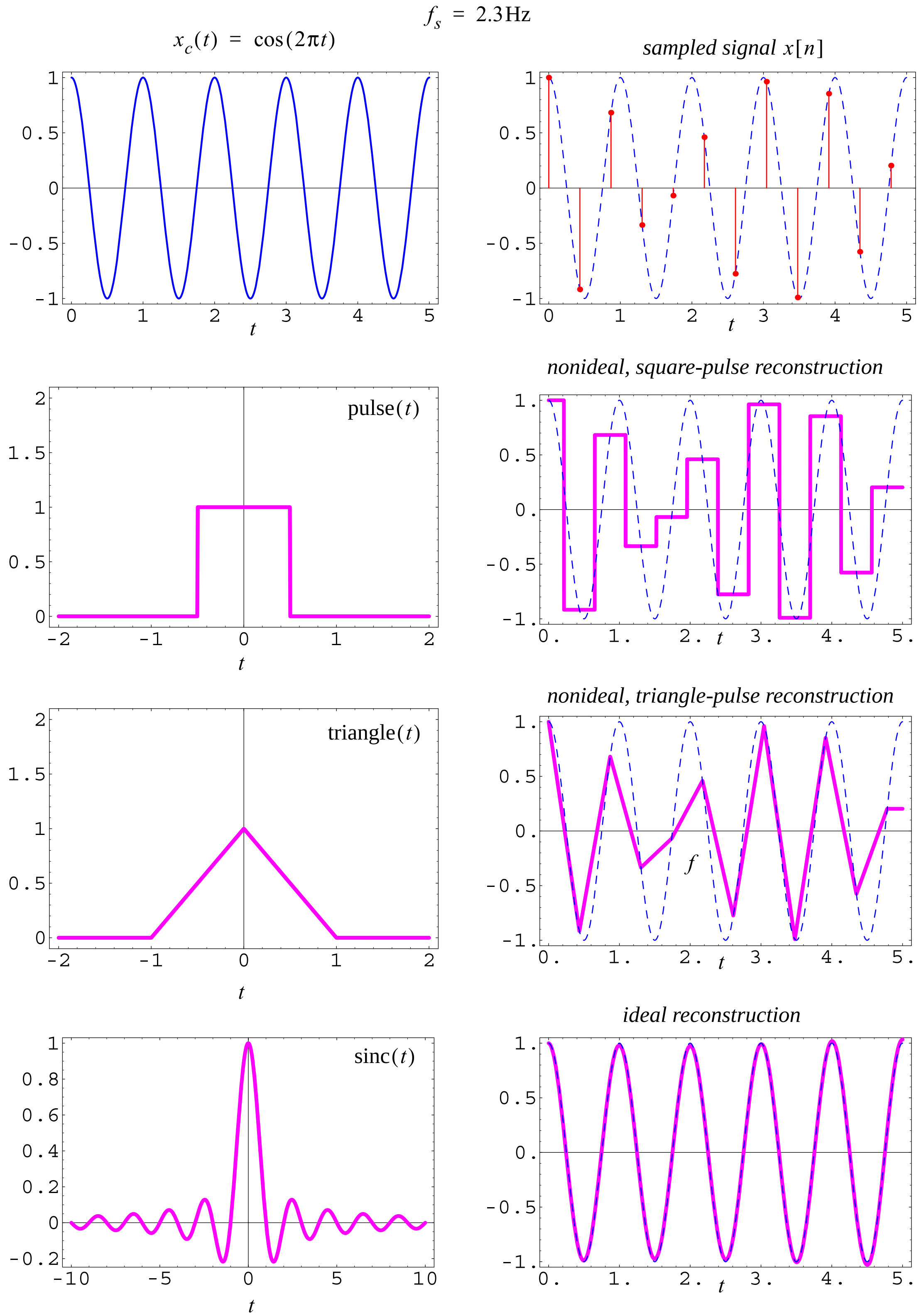

$f_s = 2.3\text{Hz}$
$x_c(t) = \cos(2\pi t)$
sampled signal $x[n]$
nonideal, square-pulse reconstruction
pulse(t)
nonideal, triangle-pulse reconstruction
triangle(t)
ideal reconstruction
sinc(t)
t

Figure 5

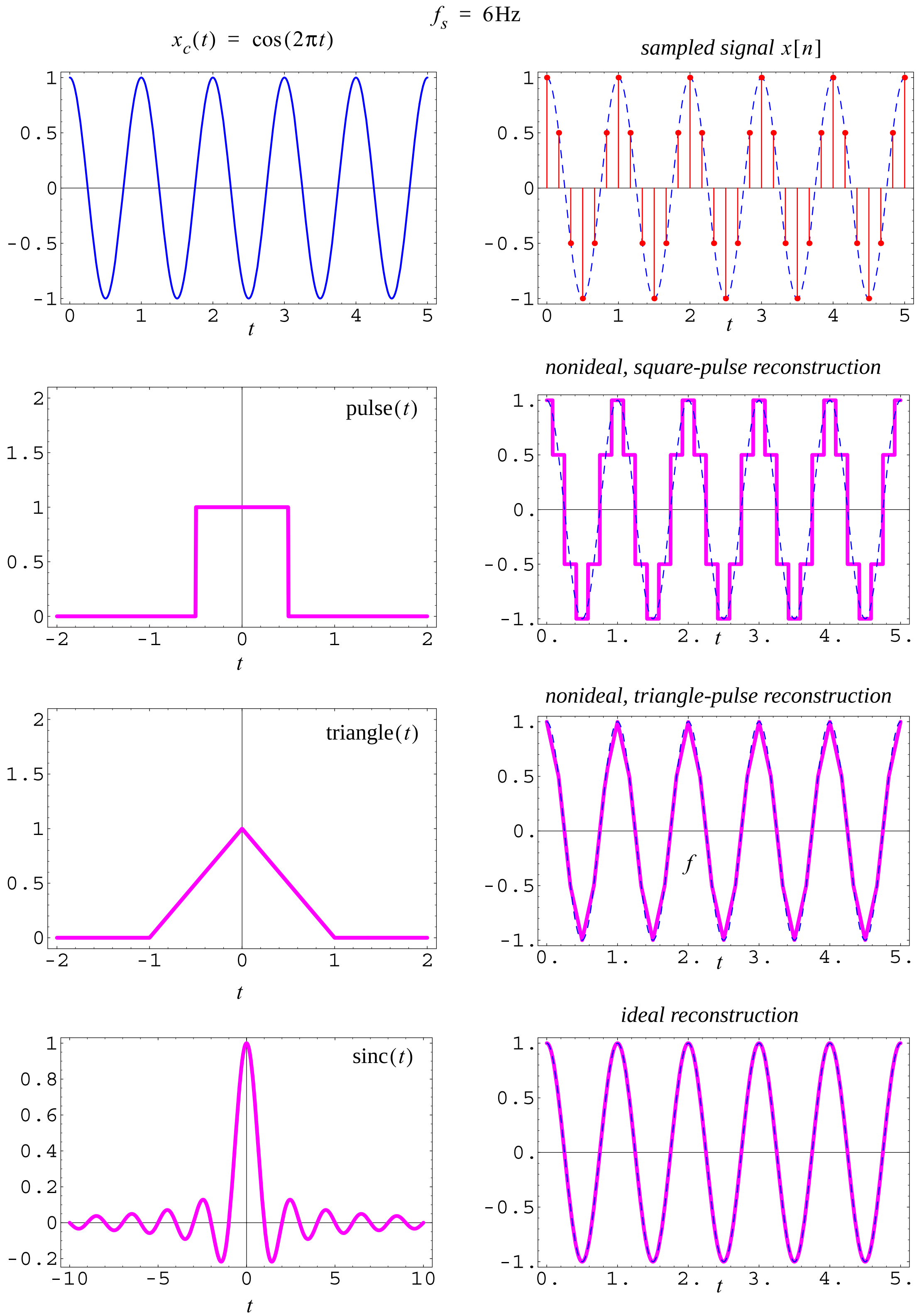
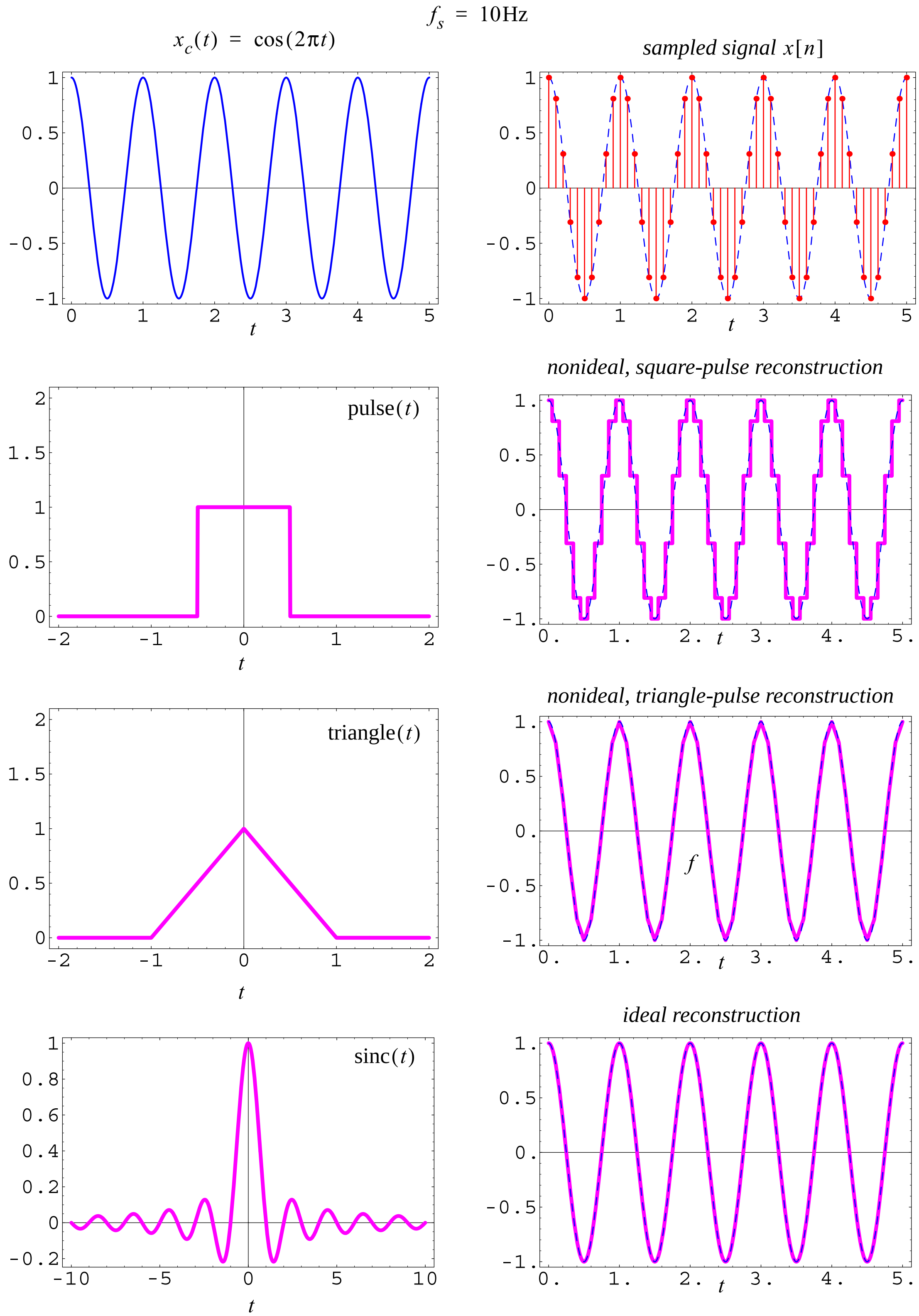

$f_s = 6\text{Hz}$
$x_c(t) = \cos(2\pi t)$
sampled signal $x[n]$
nonideal, square-pulse reconstruction
pulse(t)
nonideal, triangle-pulse reconstruction
triangle(t)
ideal reconstruction
sinc(t)
t

Figure 6



$f_s = 10\text{Hz}$
$x_c(t) = \cos(2\pi t)$
sampled signal $x[n]$
pulse(t)
nonideal, square-pulse reconstruction
triangle(t)
nonideal, triangle-pulse reconstruction
sinc(t)
ideal reconstruction
t

Figure 7