# The Discrete-Time Fourier Transform (DTFT)

## 1. Introduction

In these notes, we introduce the *discrete-time Fourier transform* (*DTFT*) and explore some of its properties.

## 2. Discrete-Time Fourier Transform (DTFT)

### A. Introduction

Previously, we have defined the *continuous-time Fourier transform* (*CTFT*) as,

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-\mathbf{j}2\pi ft}dt \tag{1}$$

where $x(t)$ is a continuous-time signal, $X(f)$ is the CTFT of the continuous-time signal, $t$ denotes the time variable (in seconds), and $f$ denotes the frequency variable (in Hertz). The *discrete-time Fourier transform* (*DTFT*) is defined by,

$$X(e^{\mathbf{j}\theta}) = \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}n\theta} \tag{2}$$

where $x[n]$ is a discrete-time signal, $X(e^{\mathbf{j}\theta})$ is the DTFT of the discrete-time signal, $n$ denotes the time index, and $\theta$ denotes the frequency variable.

Comparing the definitions for the CTFT and the DTFT, we observe the following differences. For the DTFT, the integration over time $t$ has been replaced by a summation over the time index $n$; also, the frequency variable $\theta$ is unitless. In order to relate the frequency variable $\theta$ to a real frequency, we need to know the sampling frequency $f_s$. That is, if the discrete-time sequence $x[n]$ represents a sampled continuous-time signal, sampled at frequency $f_s$, then $\theta$ and the corresponding real frequency $f$ are related as follows:

$$f = \frac{\theta f_s}{2\pi},\ \theta = \frac{2\pi f}{f_s}. \tag{3}$$

### B. Analytic example

We begin our study of the DTFT by looking at the DTFT of a simple discrete-time signal, namely a pulse of width $2M+1$ centered at $n = 0$:

$$x[n] = \begin{cases} 1 & |n| \le M \\ 0 & |n| > M \end{cases} \tag{4}$$

While in general it is much more difficult to derive an analytic expression for the DTFT than for the CTFT, for the example signal in (4) it is possible to derive a closed-form expression. Applying definition (2),

$$X(e^{\mathbf{j}\theta}) = \sum_{n=-M}^{M} e^{-\mathbf{j}n\theta} \tag{5}$$

$$X(e^{\mathbf{j}\theta}) = \sum_{n=0}^{2M} e^{-\mathbf{j}(n-M)\theta} \tag{6}$$

$$X(e^{\mathbf{j}\theta}) = e^{\mathbf{j}M\theta}\sum_{n=0}^{2M} e^{-\mathbf{j}n\theta} \tag{7}$$

$$X(e^{\mathbf{j}\theta}) = e^{\mathbf{j}M\theta}\left(\frac{1 - e^{-\mathbf{j}(2M+1)\theta}}{1 - e^{-\mathbf{j}\theta}}\right) \tag{8}$$

From (7) to (8) above, we used the following finite geometric sum identity:

$$\sum_{n=0}^{N} \alpha^n = \frac{1 - \alpha^{(N+1)}}{1 - \alpha},\ \alpha \neq 1\,. \tag{9}$$ [1]

where, for our case,

$$\alpha = e^{-\mathbf{j}\theta} \text{ and } N = 2M\,. \tag{10}$$

Equation (8) can be further simplified by symmetrizing the powers of the exponentials in the numerator and denominator:

$$X(e^{\mathbf{j}\theta}) = e^{\mathbf{j}M\theta}\left[\frac{e^{-\mathbf{j}(2M+1)\theta/2}(e^{\mathbf{j}(2M+1)\theta/2} - e^{-\mathbf{j}(2M+1)\theta/2})}{e^{-\mathbf{j}\theta/2}(e^{\mathbf{j}\theta/2} - e^{-\mathbf{j}\theta/2})}\right] \tag{11}$$

$$X(e^{\mathbf{j}\theta}) = e^{\mathbf{j}M\theta}\left(\frac{e^{-\mathbf{j}(2M+1)\theta/2}}{e^{-\mathbf{j}\theta/2}}\right)\left(\frac{e^{\mathbf{j}(2M+1)\theta/2} - e^{-\mathbf{j}(2M+1)\theta/2}}{e^{\mathbf{j}\theta/2} - e^{-\mathbf{j}\theta/2}}\right) \tag{12}$$

Note that,

$$e^{\mathbf{j}M\theta}\left(\frac{e^{-\mathbf{j}(2M+1)\theta/2}}{e^{-\mathbf{j}\theta/2}}\right) = 1 \tag{13}$$

so that equation (12) reduces to:

$$X(e^{\mathbf{j}\theta}) = \frac{e^{\mathbf{j}(2M+1)\theta/2} - e^{-\mathbf{j}(2M+1)\theta/2}}{e^{\mathbf{j}\theta/2} - e^{-\mathbf{j}\theta/2}} = \left(\frac{e^{\mathbf{j}(2M+1)\theta/2} - e^{-\mathbf{j}(2M+1)\theta/2}}{2\mathbf{j}}\right)\left(\frac{2\mathbf{j}}{e^{\mathbf{j}\theta/2} - e^{-\mathbf{j}\theta/2}}\right) \tag{14}$$

$$X(e^{\mathbf{j}\theta}) = \frac{\sin\left(\frac{\theta}{2}(2M+1)\right)}{\sin\left(\frac{\theta}{2}\right)} \tag{15}$$

In Figure 1 below, we plot $X(e^{\mathbf{j}\theta})$ for a pulse $x[n]$ with $M = 4$. Note that this DTFT appears to be periodic in the frequency variable $\theta$ with period $T = 2\pi$. This is not only the case for this example, but for the DTFT of any discrete-time signal $x[n]$. The periodic property of the DTFT is easily shown by going back to the definition in equation (2):

$$X(e^{\mathbf{j}(\theta+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}n(\theta+2\pi)} = e^{-\mathbf{j}2\pi n}\sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}n\theta} = e^{-\mathbf{j}2\pi n}X(e^{\mathbf{j}\theta}) \tag{16}$$

Note that for integer $n$,

$$e^{-\mathbf{j}2\pi n} = 1 \tag{17}$$

so that:

$$X(e^{\mathbf{j}(\theta+2\pi)}) = X(e^{\mathbf{j}\theta})\,. \tag{18}$$

---

1. *Strictly speaking, equation (9) is not valid for* $\theta = \pm 2n\pi$, $n \in \{0, 1, 2, \ldots\}$, *since* $\alpha = 1$ *for those values of* $\theta$. *For these values of* $\theta$, $X(e^{\mathbf{j}\theta}) = 2M+1$. *Using L'Hopital's Rule, this is, however, precisely the limiting value of the final expression for* $X(e^{\mathbf{j}\theta})$ *in equation (15). Therefore, we do not treat this case separately.*

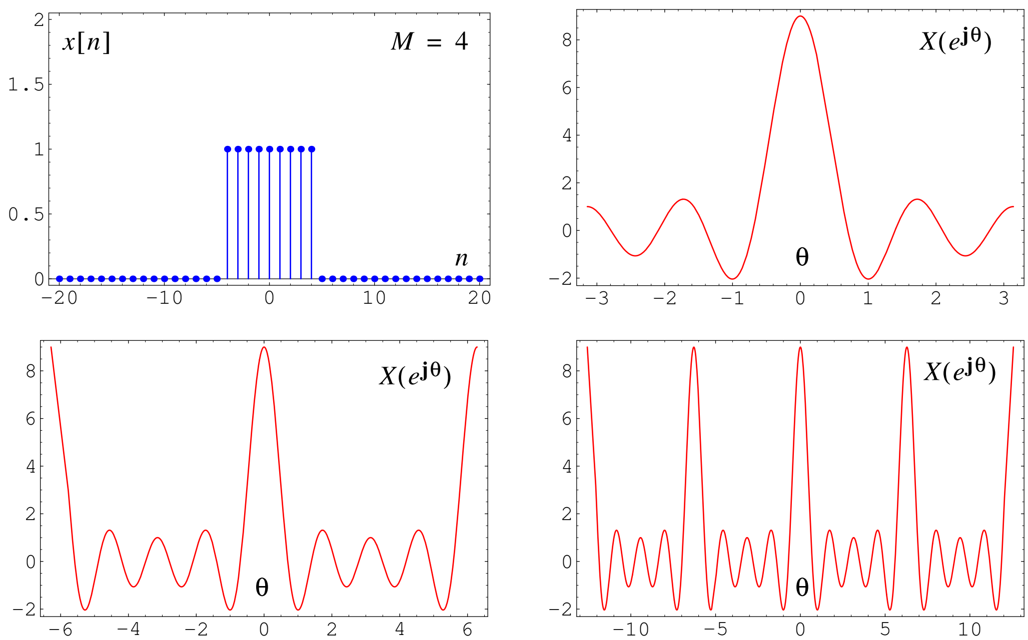


***Figure 1***

Hence, the DTFT of any discrete-time signal is periodic in $\theta$ wit period $T = 2\pi$. This result is closely related to our previous discussion on sampling and aliasing as we shall see shortly.

Now, let us see how the DTFT in equation (15) changes as a function of the pulse width parameter $M$. In Figure 2, we plot $x[n]$ and $X(e^{\mathbf{j}\theta})$ for $M = 0, 1, 2, 4, 8$. Note the strong similarity of Figure 2 below, and Figure 5 on page 7 of the *Fourier Series to Fourier Transform* notes. For the narrowest pulse ($M = 0$, $x[n] = \delta[n]$), the frequency content is uniformly distributed for $-\pi \le \theta \le \pi$, while for the wider pulses, the frequency content becomes more and more concentrated around $\theta = 0$. This is the same phenomenon as for the CTFT (Figure 5, *Fourier Series to Fourier Transform* notes).

## 3. Some additional DTFT illustrations

Below, we explore additional examples of the DTFT.

### A. Finite-length cosine wave

Here we consider the DTFT of a discrete-time signal $x[n]$, sampled from the continuous-time signal $x_c(t)$,

$$x_c(t) = \cos(2\pi t) \tag{19}$$

with sampling frequency $f_s = 10\text{Hz}$ for a total of 50 samples, such that,

$$x[n] = \begin{cases} x_c(n/f_s) & n \in \{0, 1, \ldots, 48, 49\} \\ 0 & elsewhere \end{cases}. \tag{20}$$

Applying definition (2),

$$X(e^{\mathbf{j}\theta}) = \sum_{n=0}^{49} x_c(n/f_s) e^{-\mathbf{j}n\theta} \tag{21}$$

In Figure 3, we plot $|X(e^{\mathbf{j}\theta})|$ and $\angle X(e^{\mathbf{j}\theta})$ for the discrete-time sequence in (20) as a function of the dimensionless frequency variable $\theta$, and as a function of the real frequency variable $f$ (in Hertz), where we make

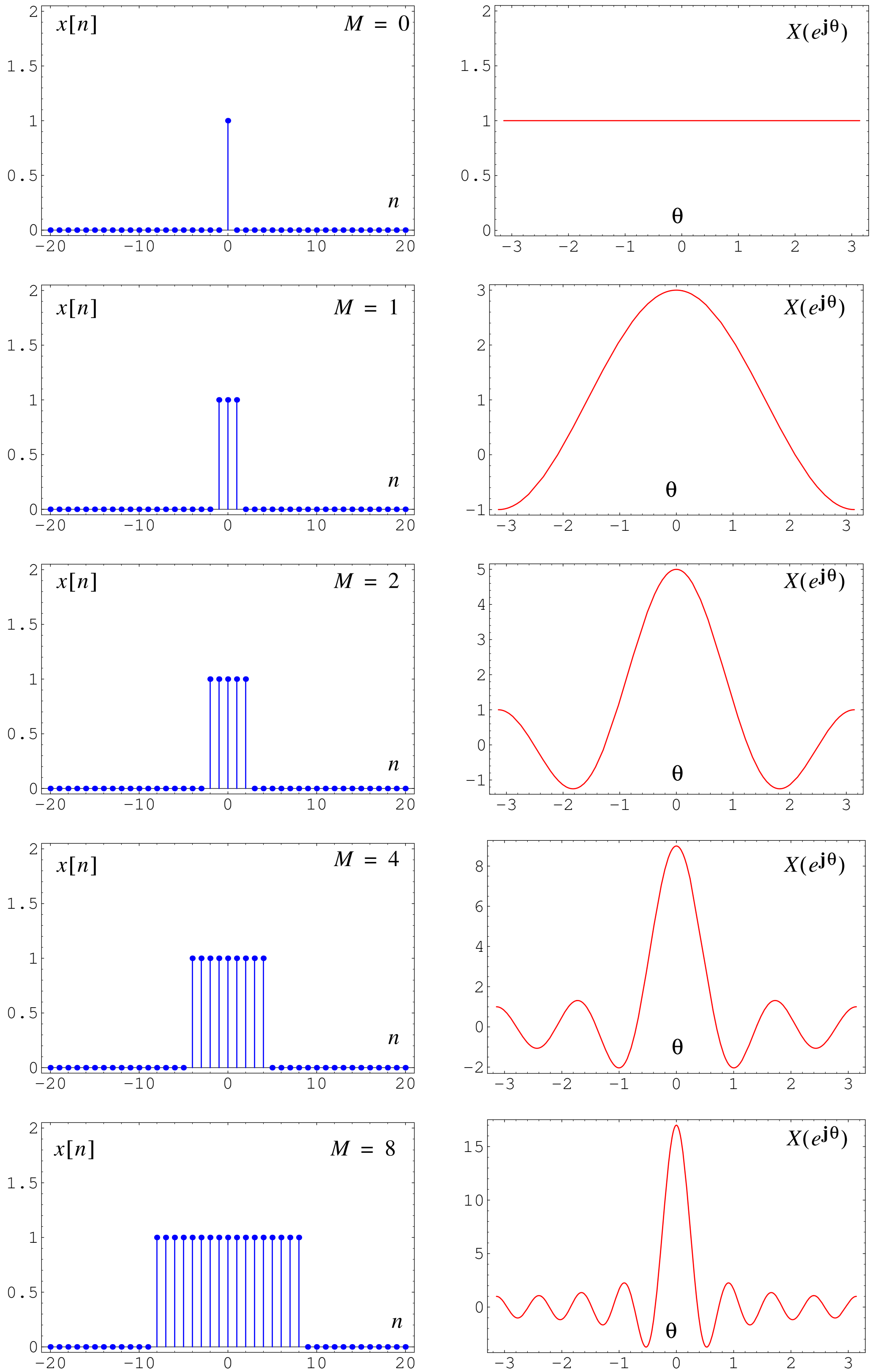


***Figure 2***

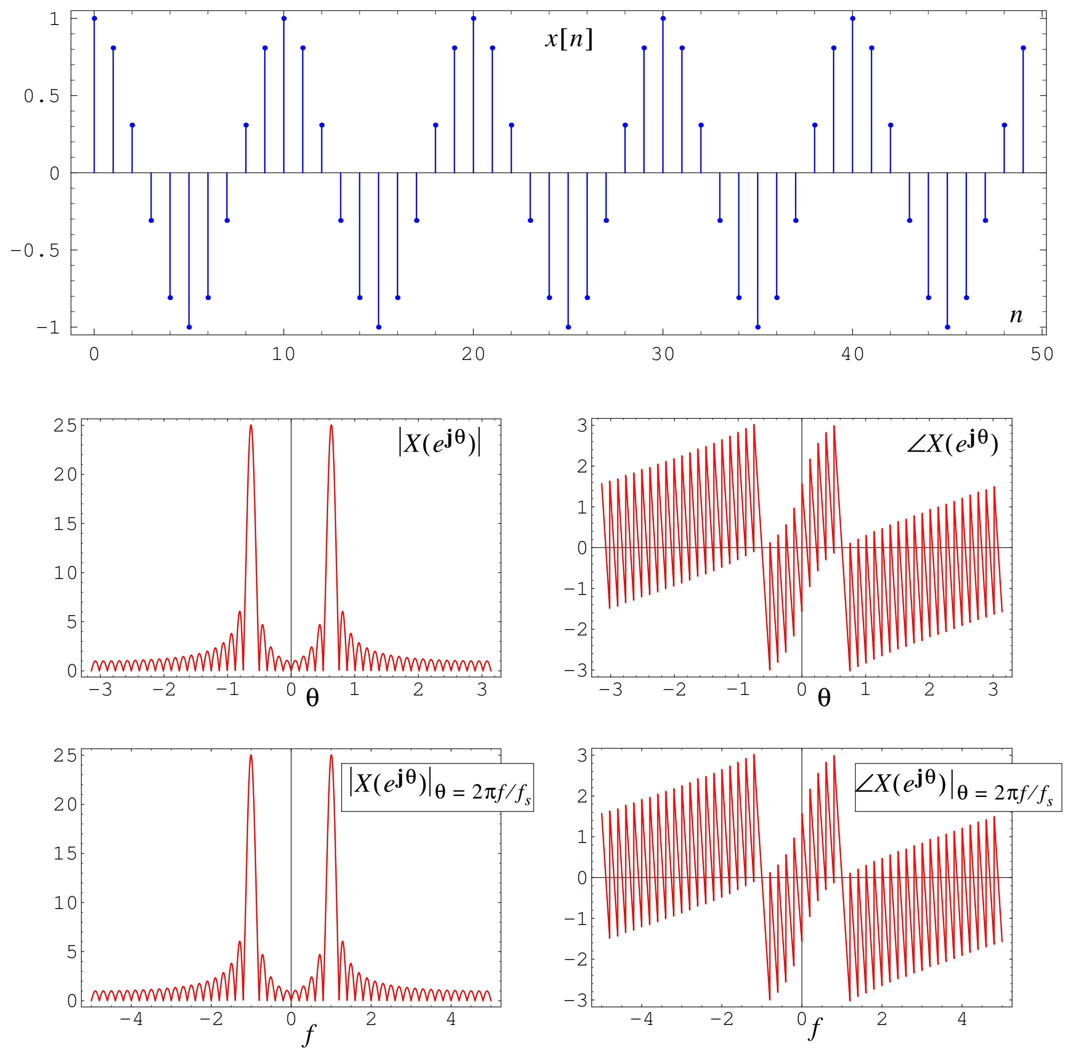


***Figure 3***

the substitution in equation (3). That is, the bottom two plots in Figure 3 correspond to the two expressions below:

$$|X(e^{j\theta})|_{\theta = 2\pi f/f_s} \text{ and } \angle X(e^{j\theta})|_{\theta = 2\pi f/f_s} \tag{22}$$

These plots are more meaningful to us, since they clearly shows two spikes in the DTFT centered at $\pm 1\text{Hz}$. Now, let us make a few observations. First, note that when plotted as a function of $f$, one period of the DTFT covers the frequency range $[-f_s/2, f_s/2] = [-5\text{Hz}, 5\text{Hz}]$ ($f_s = 10\text{Hz}$). Since the DTFT is periodic, this same frequency spectrum will be repeated outside this frequency range. Recall from our discussion on sampling, this is exactly what we said would occur —namely, that the frequency spectrum of the original continuous-time waveform will be repeated at offsets of $\pm kf_s$, $k \in \{1, 2, 3, \ldots\}$. In Figure 4, for example, we plot $|X(e^{j\theta})|_{\theta = 2\pi f/f_s}$ for $-25\text{Hz} < f < 25\text{Hz}$, corresponding to $-5\pi < \theta < 5\pi$.

At this point, the reader might object to the above discussion on the grounds that we have previously said that the continuous-time spectrum of the cosine wave $x_c(t)$ in equation (19) is given by,

$$X_c(f) = \frac{1}{2}\delta(f+1) + \frac{1}{2}\delta(f-1) \tag{23}$$

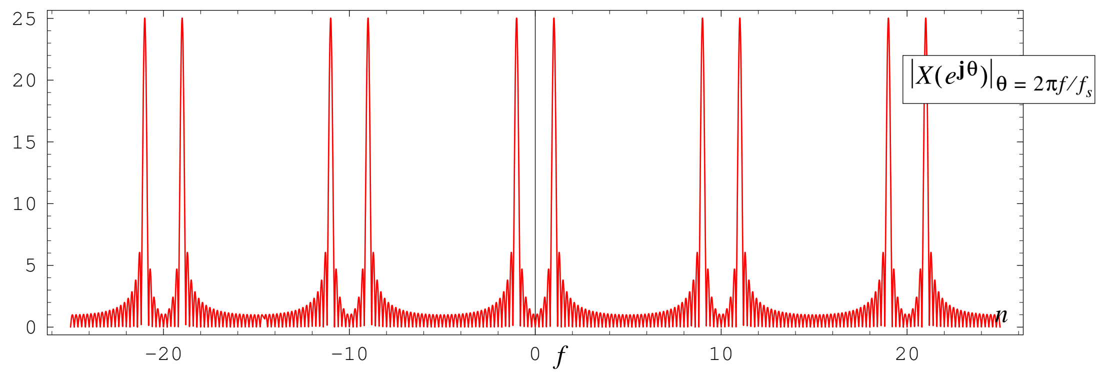


***Figure 4***

That is, the CTFT has two distinct spikes at $\pm 1\text{Hz}$, and is zero everywhere else. The CTFT in (23) is, however, correct only for a cosine wave that is not time-limited. Note that by restricting our sampling of the cosine wave to 5 periods of the 1Hz cosine wave (50 samples), we implicitly assumed that the discrete-time signal $x[n]$ is zero everywhere else. That is, the DTFTs in Figures 3 and 4 represent the discrete-time frequency spectrum of a *time-limited* cosine wave. For comparison, we can compute the CTFT for the following time-limited continuous-time function,

$$x(t) = \cos(2\pi t)[u(t) - u(t-5)] \tag{24}$$

by applying definition (1):

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-\mathbf{j}2\pi f t} dt = \int_0^5 \cos(2\pi t) e^{-\mathbf{j}2\pi f t} dt \tag{25}$$

Using the inverse Euler relations for the cosine function:

$$X(f) = \int_0^5 \left(\frac{1}{2}e^{\mathbf{j}2\pi t} + \frac{1}{2}e^{-\mathbf{j}2\pi t}\right) e^{-\mathbf{j}2\pi f t} dt \tag{26}$$

$$X(f) = \frac{1}{2}\int_0^5 [e^{-\mathbf{j}2\pi t(f-1)} + e^{-\mathbf{j}2\pi t(f+1)}] dt \tag{27}$$

$$X(f) = \frac{1}{2}\left[\frac{e^{-\mathbf{j}2\pi t(f-1)}}{-\mathbf{j}2\pi(f-1)} + \frac{e^{-\mathbf{j}2\pi t(f+1)}}{-\mathbf{j}2\pi(f+1)}\right]_{t=0}^{t=5} \tag{28}$$

$$X(f) = \frac{1}{2}\left[\left(\frac{e^{-\mathbf{j}10\pi(f-1)}}{-\mathbf{j}2\pi(f-1)} + \frac{e^{-\mathbf{j}10\pi(f+1)}}{-\mathbf{j}2\pi(f-1)}\right) - \left(\frac{1}{-\mathbf{j}2\pi(f-1)} + \frac{1}{-\mathbf{j}2\pi(f-1)}\right)\right] \tag{29}$$

$$X(f) = \frac{\mathbf{j}}{2}\left(\frac{e^{-\mathbf{j}10\pi(f-1)} - 1}{2\pi(f-1)} + \frac{e^{-\mathbf{j}10\pi(f+1)} - 1}{2\pi(f-1)}\right) \tag{30}$$

Figure 5 below plots the time-limited, continuous-time signal in (24) and its CTFT magnitude spectrum $|X(f)|$. Note that except for a scaling difference, the CTFT and the DTFT appear very similar over the frequency range $[-f_s/2, f_s/2]$. Thus, the fact that the DTFT spectral representation of the sampled, time-limited cosine wave is not entirely localized at $\pm 1\text{Hz}$ (as might have been expected) is not a consequence of the DTFT itself, but rather a consequence of the *finite-length* sampling process. We refer to this phenomenon, namely, the spreading of frequency content from the idealized peaks to the entire frequency range due to finite-length sampling, as *spectral leakage*.

The longer the sequence length, the more concentrated the frequency content of that discrete-time signal will be in the neighborhood of its dominant frequencies. Below, we consider the DTFT for the discrete-time signal $x[n]$,

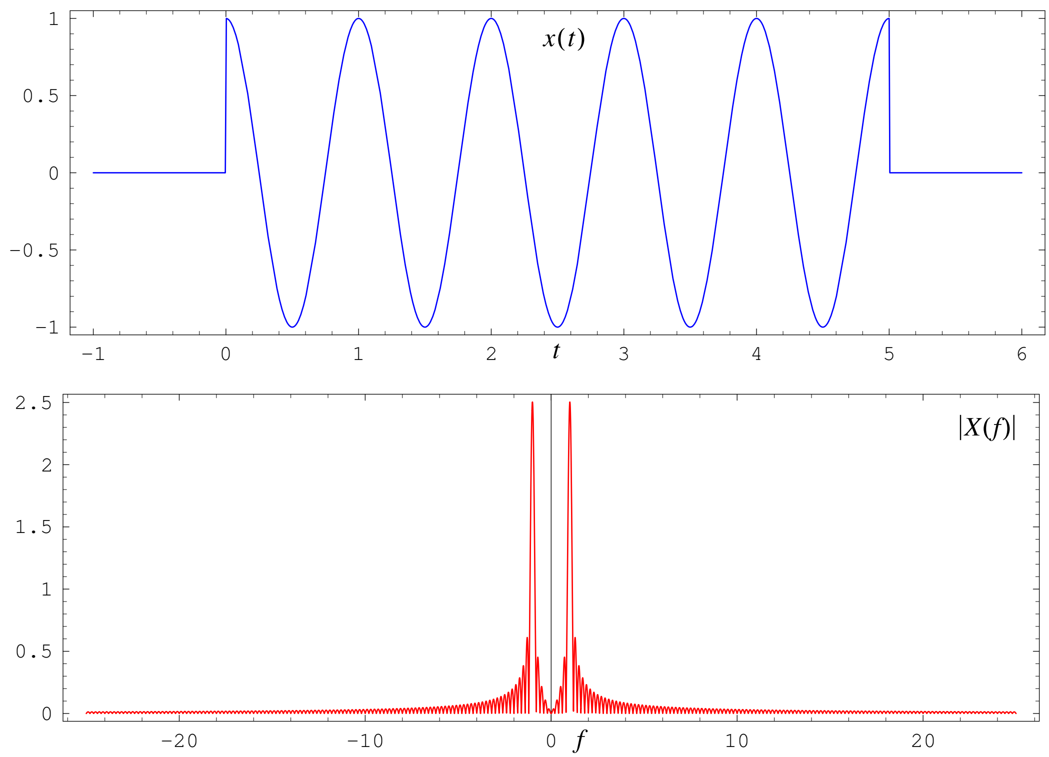


**Figure 5**

$$x[n] = \begin{cases} x_c(n/f_s) & n \in \{0, 1, \ldots, 198, 199\} \\ 0 & elsewhere \end{cases} \tag{31}$$

where $x_c(t)$ is again a 1Hz cosine wave, and the sampling frequency is again given by $f_s = 10\text{Hz}$. Note that the difference between the discrete-time signals in equations (31) and (20) is that $x[n]$ now consists of 200 samples (20 cycles of the cosine wave), instead of 50 samples (5 cycles of the cosine wave). In Figure 6, we plot the DTFT as a function of frequency $f$ for $x[n]$ in equation (31). Comparing the plots in Figure 6 to those in Figure 3 (50 samples), note how much more tightly focused the frequency content is about the frequencies $\pm 1\text{Hz}$.

### B. Finite-length sum of cosines

Here we consider the DTFT of a discrete-time signal $x[n]$, sampled from the continuous-time signal $x_c(t)$,

$$x_c(t) = 1 + 2\cos(2\pi t) + 4\cos(4\pi t) \tag{32}$$

with sampling frequency $f_s = 10\text{Hz}$ for a total of 50 samples, such that,

$$x[n] = \begin{cases} x_c(n/f_s) & n \in \{0, 1, \ldots, 48, 49\} \\ 0 & elsewhere \end{cases}. \tag{33}$$

Applying definition (2),

$$X(e^{\mathbf{j}\theta}) = \sum_{n=0}^{49} x_c(n/f_s) e^{-\mathbf{j}n\theta} \tag{34}$$

In Figure 7, we plot $|X(e^{\mathbf{j}\theta})|$ and $\angle X(e^{\mathbf{j}\theta})$ for the discrete-time sequence in (33) as a function of the dimensionless frequency variable $\theta$, and as a function of the real frequency variable $f$ (in Hertz), where we again

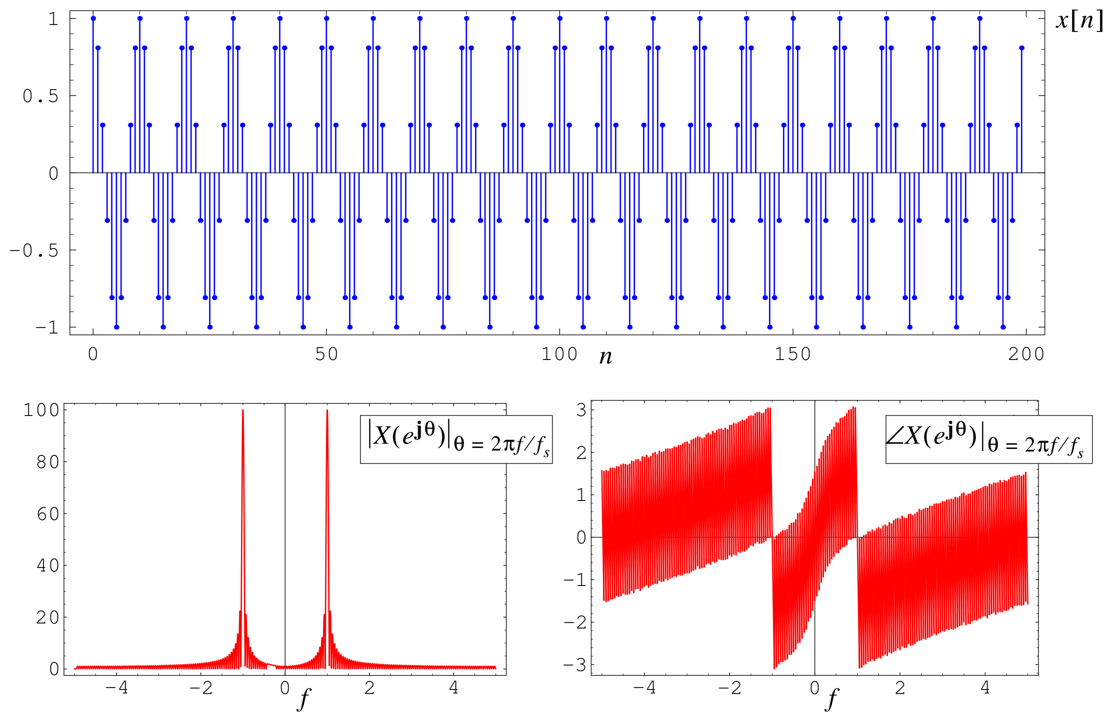


***Figure 6***

make the substitution in equation (3). That is, the bottom two plots in Figure 7 correspond to the two expressions below:

$$|X(e^{\mathbf{j}\theta})|_{\theta = 2\pi f/f_s} \text{ and } \angle X(e^{\mathbf{j}\theta})|_{\theta = 2\pi f/f_s} \tag{35}$$

Note the peaks in the frequency spectrum at $f = 0\text{Hz}$, $f = \pm 1\text{Hz}$ and $f = \pm 2\text{Hz}$, corresponding to the three terms in equation (32). In Figure 8, we plot $|X(e^{\mathbf{j}\theta})|_{\theta = 2\pi f/f_s}$ for $-25\text{Hz} < f < 25\text{Hz}$, corresponding to $-5\pi < \theta < 5\pi$. Note again that the frequency spectrum of the time-limited continuous-time waveform is replicated at offsets of $\pm k f_s$, $k \in \{1, 2, 3, \ldots\}$ for the time-limited sampled waveform. Note the similarity of the spectrum in Figure 8 to that of the idealized, infinite-time sampled waveform, plotted in Figure 9 below.

In the examples above, the sampling frequency $f_s$ was larger than the Nyquist frequency of $2f_{max}$. In the next section, we will look at the DTFT when the sampling frequency is less than the Nyquist frequency.

## 4. Aliasing and the Discrete-Time Fourier Transform (DTFT)

### A. Introduction

In the previous section, we saw an example of the discrete-time Fourier Transform (DTFT) for finite-length sequences $x[n]$,

$$x[n] = \begin{cases} x_c(n/f_s) & n \in \{0, 1, \ldots, 48, 49\} \\ 0 & elsewhere \end{cases} \tag{36}$$

where,

$$x_c(t) = 1 + 2\cos(2\pi t) + 4\cos(4\pi t) \tag{37}$$

and the sampling frequency is given as $f_s = 10\text{Hz}$, which is greater than the Nyquist frequency,

$$2f_{max} = 2 \times 2\text{Hz} = 4\text{Hz}. \tag{38}$$

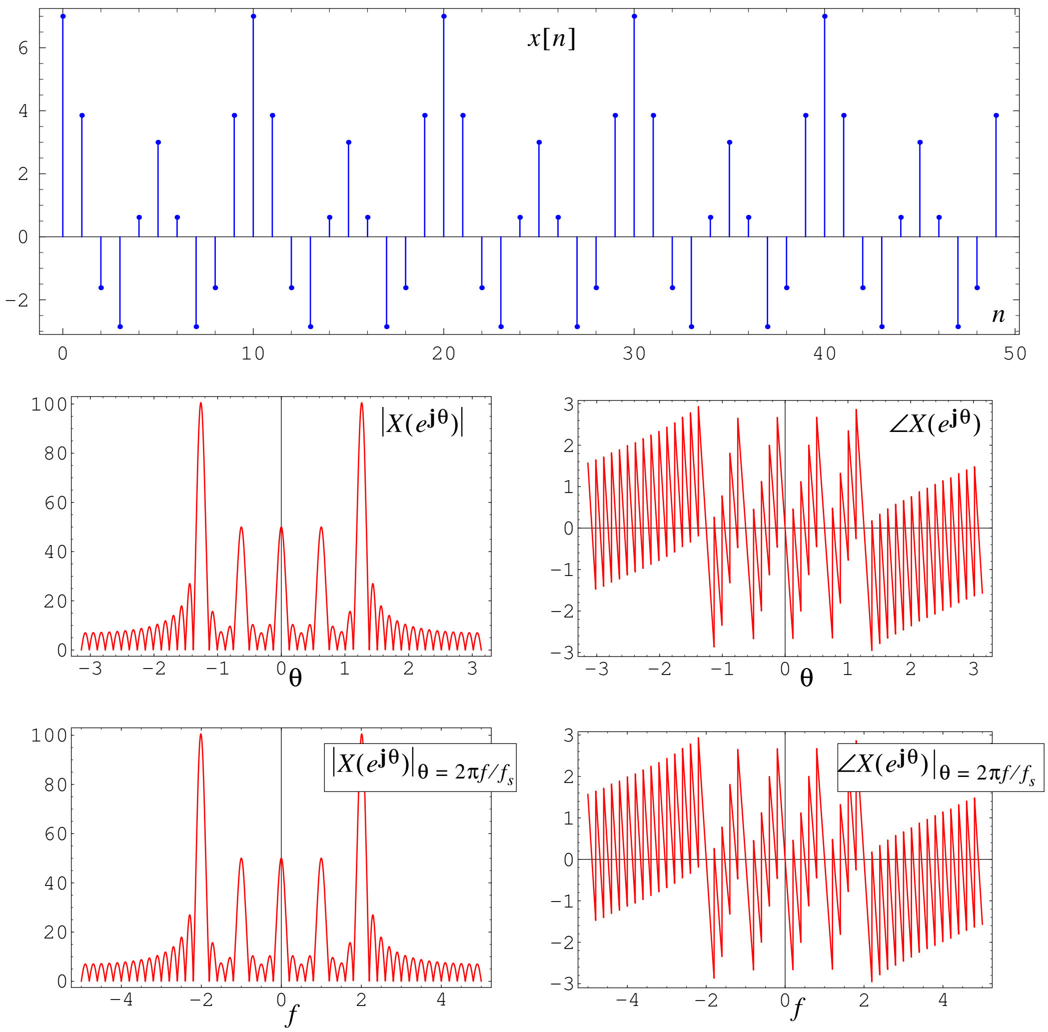


***Figure 7***

Below, we explore the DTFT for the same continuous-time signal, but for two sampling frequencies below the Nyquist frequency.

## B. Undersampled examples

Here we look at the DTFT of the following finite-length sequence:

$$x[n] = \begin{cases} x_c(n/f_s) & n \in \{0, 1, \ldots, 14, 15\} \\ 0 & elsewhere \end{cases} \tag{39}$$

where $f_s = 3\text{Hz}$ and $x_c(t)$ is given by equation (37) above. That is, we sample the continuous-time signal $x_c(t)$ for the same length of time, only at a lower sampling frequency (below the Nyquist frequency of 4Hz) than before. In order to relate our present discussion to our previous discussion on sampling and aliasing, we plot the following functions in Figure 10 below: (1) the original signal $x_c(t)$; (2) the sampled signal $x[n]$; (3) the infinite-length, continuous-time frequency spectrum $X_c(f)$ (CTFT) of the continuous-time signal $x_c(t)$; (4) the sampled freq. spectrum $X_s(f)$ (CTFT) for the infinite-length, continuous-time signal $x_s(t)$,

$$x_s(t) = \sum_{n=-\infty}^{\infty} x'[n]\delta\left(t - \frac{n}{f_s}\right) \tag{40}$$

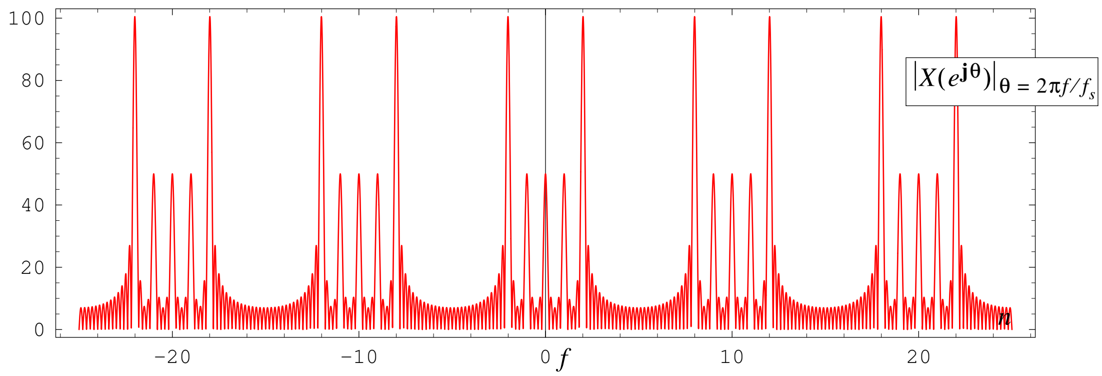


***Figure 8***

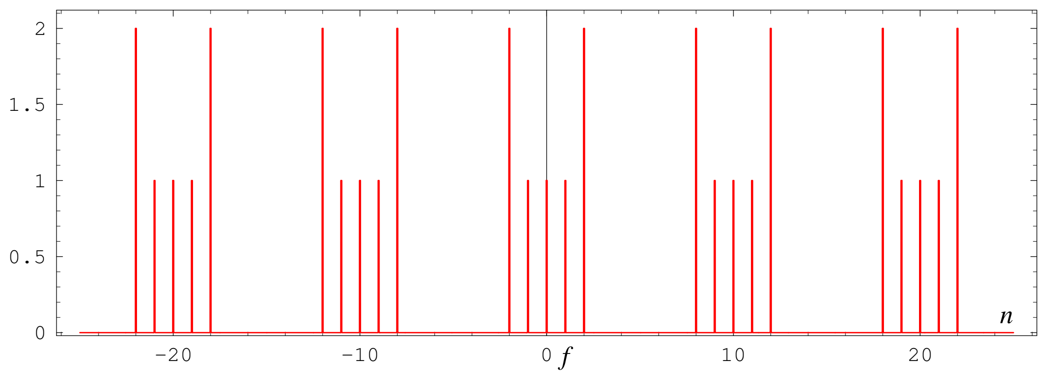


***Figure 9: Sampled spectrum of infinite-length sampled sum of cosines.***

where $x'[n] = x_c(n/f_s)$, $\forall n$; and (5) the magnitude DTFT $|X(e^{j\theta})|_{\theta = 2\pi f/f_s}$ of the finite-length, discrete-time signal $x[n]$ as a function of frequency $f$. Note that except for the spectral leakage (defined previously) caused by the finite-length of $x[n]$, the DTFT has peaks of the same relative magnitude and at the same frequencies as $X_s(f)$.

In Figure 11, we plot the same functions as in Figure 10, except that now the finite-length sequence is given by:

$$x[n] = \begin{cases} x_c(n/f_s) & n \in \{0, 1, \ldots, 16, 17\} \\ 0 & elsewhere \end{cases} \tag{41}$$

where $f_s = 3.5\text{Hz}$ and $x_c(t)$ is again given by equation (37) above. Again, note the similarity between the DTFT and $X_s(f)$.

### C. Oversampled example

Finally for comparison, we generate the same five plots as above for the oversampled case from the previous section (Figure 12). The finite-length sequence is now the same as last time [equation (36)] with $f_s = 10\text{Hz}$. In all three of these examples, the DTFT for the finite-length sampled sequences generates a similar frequency distribution as the CTFT for the infinite-length sampled sequences (represented in the continuous-time domain as $x_s(t)$), with two main differences: (1) scaling and (2) spectral leakage caused by the finite-length sampling processes in equations (36), (39) and (41).

## 5. Conclusion

The *Mathematica* notebook "dtft.nb" was used to generate the examples in this set of notes. Next time, we will introduce the *discrete Fourier transform (DFT)* and show how it is related to the DTFT.

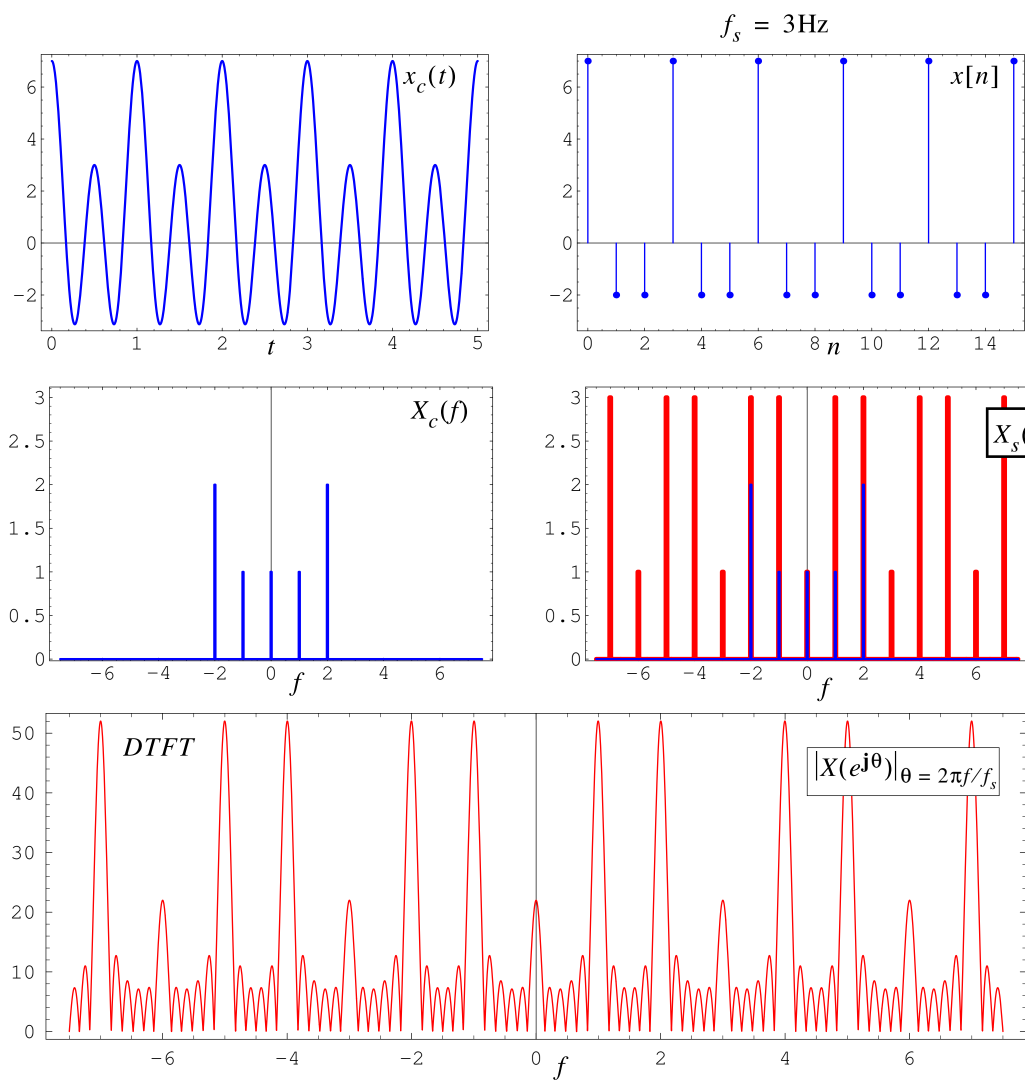


***Figure 10***

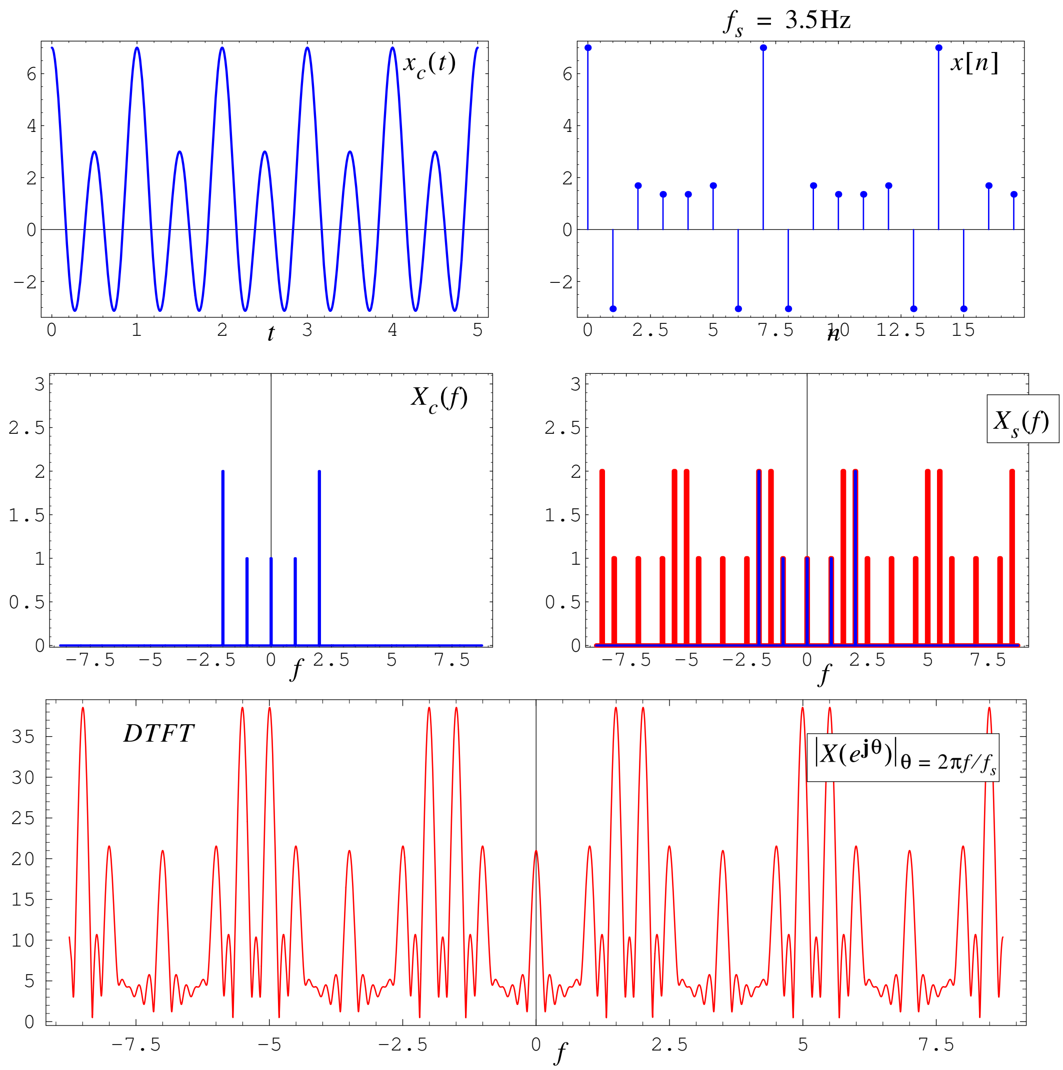
$f_s = 3.5\text{Hz}$
$x_c(t)$
$t$
$x[n]$
$n$
$X_c(f)$
$f$
$X_s(f)$
$f$
DTFT
$|X(e^{\mathbf{j}\theta})|_{\theta = 2\pi f/f_s}$
$f$

***Figure 11***

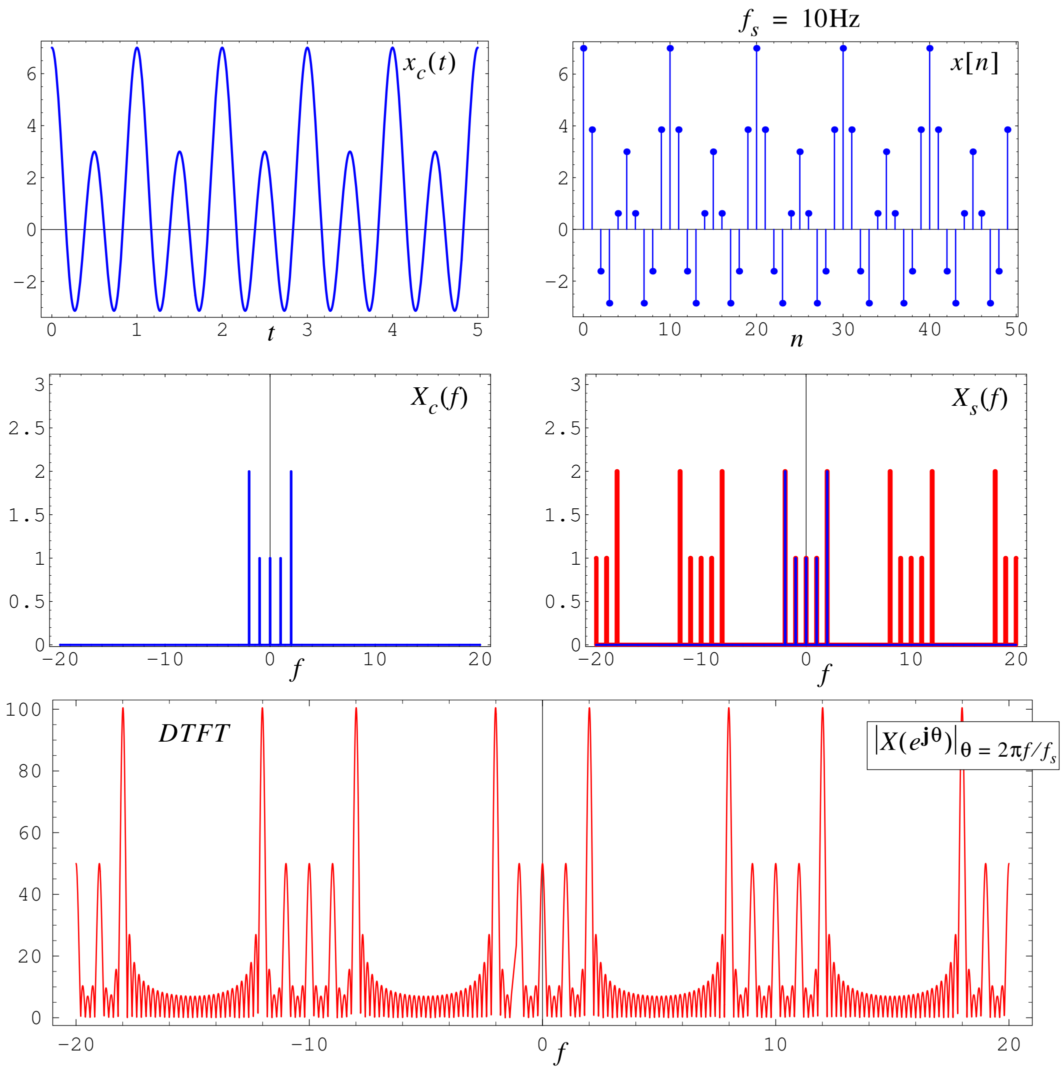


***Figure 12***