EEL6825: Homework #1 (Fall 2003)

(4 problems, distributed 9/7/2003, due 9/18/2003)

Note: Submit ***detailed*** and complete solutions to the following problems, including well commented programming code that you use for any of your solutions.

Problem 1: (10 points)

Assume that two-dimensional random vectors $\mathbf{x}$ and $\mathbf{y}$, defined by,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{1}$$

are characterized by the following probability density functions:

$$p(\mathbf{x}) = \begin{cases} k_x & 2 \le x_1 \le 3 \text{ and } 0 \le x_2 \le 3 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$p(\mathbf{y}) = \begin{cases} k_y e^{\frac{y_1 + 2y_2}{4}} & 0 \le y_1 \le \ln 2 \text{ and } 0 \le y_2 \le \ln 2 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Compute the following:

(a) Constants k_x and k_y. (2 points)

(b) Mean vectors $\mu_{\mathbf{x}}$ and $\mu_{\mathbf{y}}$. (4 points)

(c) Covariance matrices $\Sigma_{\mathbf{x}}$ and $\Sigma_{\mathbf{y}}$. (4 points)

Problem 2: (20 points)

During some flight experiments, the aircraft altitude a and the outside air pressure p are measured in miles and inches of mercury, respectively. Afterwards, N of these measurements are used to estimate the covariance matrix $\Sigma_{\mathbf{x}}$ for the two-dimensional random vector:

$$\mathbf{x} = \begin{bmatrix} a \\ p \end{bmatrix} \tag{4}$$

The covariance matrix $\Sigma_{\mathbf{x}}$ was computed as:

$$\Sigma_{\mathbf{x}} = \frac{1}{N}\sum_{i=1}^{N} (\mathbf{x}_i - \mu_{\mathbf{x}})(\mathbf{x}_i - \mu_{\mathbf{x}})^T \tag{5}$$

where $\mathbf{x}_i$ denotes the ith measurement, and $\mu_{\mathbf{x}}$ denotes the mean value for the N measurements of $\mathbf{x}$.

(a) Which of the following nine matrices is the most likely covariance matrix $\Sigma_{\mathbf{x}}$? Explain your reasoning. (18 points)

(b) Compute the eigenvalues and eigenvectors for your answer in part (a). (2 points)

$$\Sigma_1 = \begin{bmatrix} 1.232 & 0.0013 \\ 0.0013 & 2.791 \end{bmatrix} \qquad \Sigma_2 = \begin{bmatrix} 1.232 & 0.867 \\ -0.867 & 2.791 \end{bmatrix} \qquad \Sigma_3 = \begin{bmatrix} 1.232 & -0.867 \\ -0.867 & 3.307 \end{bmatrix}$$

$$\Sigma_4 = \begin{bmatrix} 1.232 & -0.0013 \\ -0.0013 & 2.791 \end{bmatrix} \qquad \Sigma_5 = \begin{bmatrix} 1.232 & 0.867 \\ 0.867 & 2.791 \end{bmatrix} \qquad \Sigma_6 = \begin{bmatrix} 1.232 & -3.307 \\ -3.307 & 2.791 \end{bmatrix}$$

$$\Sigma_7 = \begin{bmatrix} 1.232 & -0.812 \\ -0.812 & 2.791 \end{bmatrix} \qquad \Sigma_8 = \begin{bmatrix} 1.232 & 5.867 \\ 5.867 & 2.791 \end{bmatrix} \qquad \Sigma_9 = \begin{bmatrix} 1232.5 & -0.867 \\ -0.867 & 0.91 \end{bmatrix}$$

Problem 3: (40 points)

Two classes ω_1 and ω_2 are characterized by 2D normal distributions with the following parameters:

$$P(\omega_1) = P(\omega_2) \tag{6}$$

$$\mu_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mu_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{7}$$

Plot contour plots (20 points) and the Bayes decision boundary (20 points) for each of the following cases:

(a) $\Sigma_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \Sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\Sigma_1 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \Sigma_2 = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

Problem 4: (20 points)

You are given two images *gator* and *smilie* in different file formats (jpg, bmp, ppm). Your task is to statistically analyze these two images, using whichever file format is most convenient for you. Each pixel of an image, say *gator*, can be represented by a vector,

$$\mathbf{p}^{3D}_{gator} = [r, g, b]^T \tag{8}$$

in the RGB color space, where r denotes red, g denotes green and b denotes blue channel values. Similarly, for the image *smilie*, each pixel is represented as a 3D vector $\mathbf{p}^{3D}_{smilie}$.

Assuming that the vectors $\mathbf{p}^{3D}_{gator}$ and $\mathbf{p}^{3D}_{smilie}$ have 3D Gaussian distributions, compute the maximum likelihood (ML) estimates for the mean vectors μ^{3D}_{gator} and μ^{3D}_{smilie}, and the ML estimates for the covariance matrices Σ^{3D}_{gator} and Σ^{3D}_{smilie}.