Discriminant examples for the Normal density

Here, we look at what decision boundaries two Normal-distributed classes can form between them. Assuming equal priors $P(\omega_i)$, a decision boundary between two pdfs is given by,

$$p(\mathbf{x}|\omega_1) = p(\mathbf{x}|\omega_2) \text{ or equivalently by,} \tag{1}$$

$$\ln p(\mathbf{x}|\omega_1) = \ln p(\mathbf{x}|\omega_2) \tag{2}$$

For two Normal densities, this decision boundary is given by:

$$\frac{1}{(2\pi)^{d/2}|\Sigma_1|^{1/2}}\exp\left[-\frac{1}{2}(\mathbf{x}-\mu_1)^T\Sigma_1^{-1}(\mathbf{x}-\mu_1)\right] = \frac{1}{(2\pi)^{d/2}|\Sigma_2|^{1/2}}\exp\left[-\frac{1}{2}(\mathbf{x}-\mu_2)^T\Sigma_2^{-1}(\mathbf{x}-\mu_2)\right] \tag{3}$$

or, in log-likelihood form:

$$-\frac{1}{2}(\mathbf{x}-\mu_1)^T\Sigma_1^{-1}(\mathbf{x}-\mu_1) - \frac{1}{2}\ln|\Sigma_1| - \frac{d}{2}\ln 2\pi = -\frac{1}{2}(\mathbf{x}-\mu_2)^T\Sigma_2^{-1}(\mathbf{x}-\mu_2) - \frac{1}{2}\ln|\Sigma_2| - \frac{d}{2}\ln 2\pi \tag{4}$$

where we used the following property of the natural logarithm:

$$\ln[a(b^c)] = \ln a + c\ln b \tag{5}$$

Equation (4) can be simplified to:

$$(\mathbf{x}-\mu_1)^T\Sigma_1^{-1}(\mathbf{x}-\mu_1) + \ln|\Sigma_1| = (\mathbf{x}-\mu_2)^T\Sigma_2^{-1}(\mathbf{x}-\mu_2) + \ln|\Sigma_2| \tag{6}$$

Depending on the exact parameters of the two classes, the decision boundary in (6) represents a d-dimensional conic section (e.g. circle, ellipse, parabola, hyperbola). Let's look at a simple example.

Example: Let,

$$\mathbf{x} = (x, y) \tag{7}$$

$$\mu_1 = \mu_2 = (0, 0) \tag{8}$$

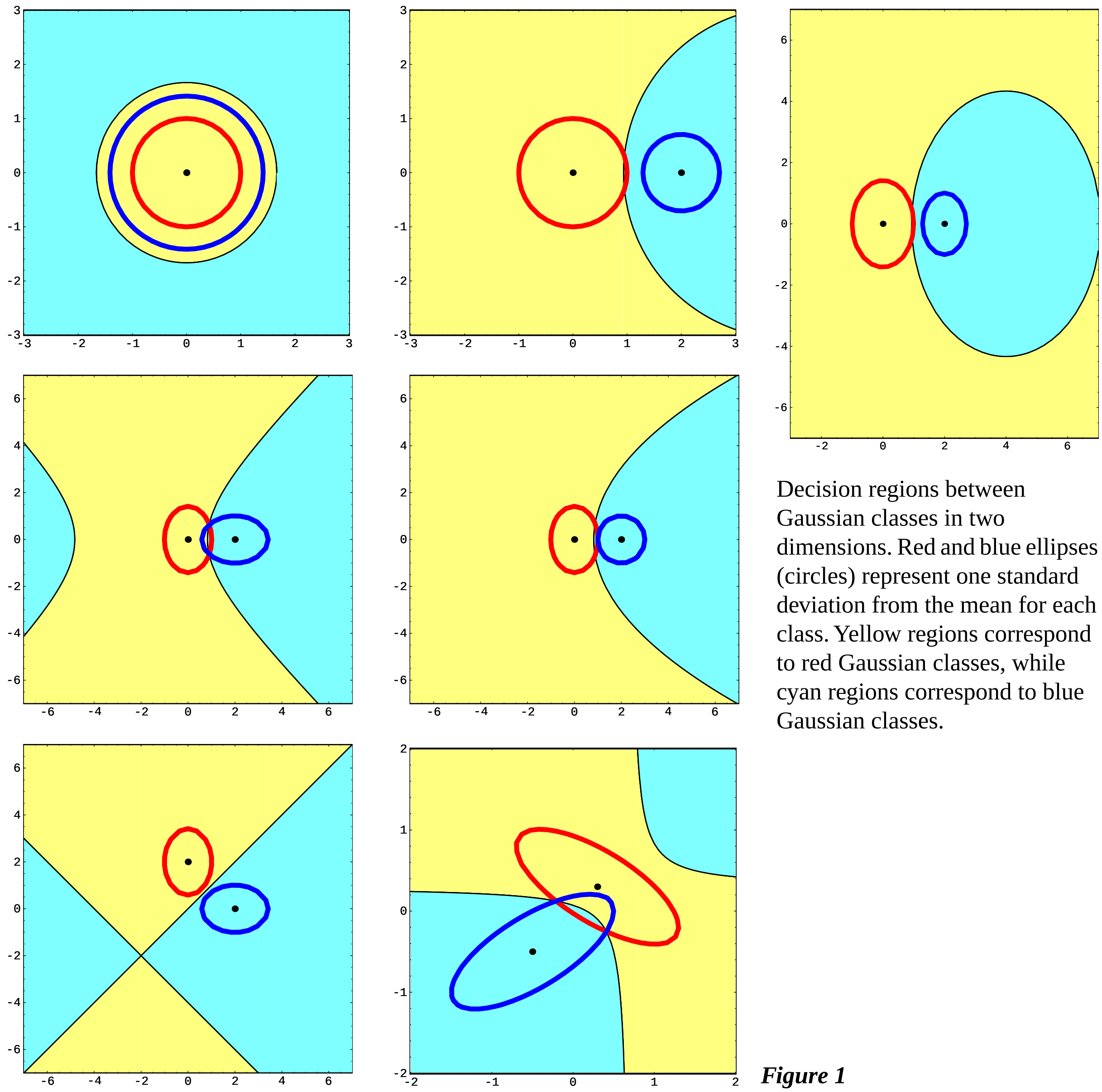
$$\Sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \Sigma_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \tag{9}$$

Substituting equations (7) through (9) into (6), we get:

$$(x^2 + y^2) = \frac{1}{2}(x^2 + y^2) + \ln(4) \tag{10}$$

$$x^2 + y^2 = 4\ln(2) \tag{11}$$

We recognize (11) as the equation of a circle with radius $2\sqrt{\ln(2)} \approx 1.665$. The region inside this circle will be classified as class ω_1, while the region outside the circle will be classified as class ω_2. Additional derivations and examples of decision boundaries may be found in the *Mathematica* notebook "normal_discriminants.nb." Figure 1 below plots decision region examples computed in that *Mathematica* notebook, including the above example (top left corner of Figure 1). Also, Figure 2 below illustrates the decision region between two three-dimensional Gaussian classes (also derived in the *Mathematica* notebook).



Decision regions between Gaussian classes in two dimensions. Red and blue ellipses (circles) represent one standard deviation from the mean for each class. Yellow regions correspond to red Gaussian classes, while cyan regions correspond to blue Gaussian classes.

Figure 1

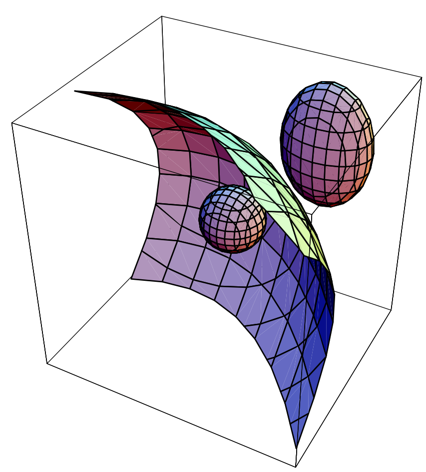


Figure 2