EEL3135: Homework #2 Solutions

Problem 1:

Consider the continuous-time signal:

$$x_1(t) = \cos(12\pi t). \tag{1}$$

Assume that you sample $x_1(t)$ such that,

$$x_1[n] = x_1(n/f_s) \tag{2}$$

where $f_s = 5$ Hz, and then pass the sampled sequence $x_1[n]$ through an ideal low-pass filter with cut-off frequencies at $\pm f_s/2 = \pm 2.5$ Hz to produce the signal $x_r(t)$.

(a) Give an analytic expression for $x_r(t)$. Be sure to explain all the steps required to get your answer.

Figure 1 illustrates the steps in deriving $x_r(t)$. The spectrum for $x_1(t)$ is given by:

$$X_6(f) = \frac{1}{2}\delta(f+6) + \frac{1}{2}\delta(f-6) \tag{S-1}$$

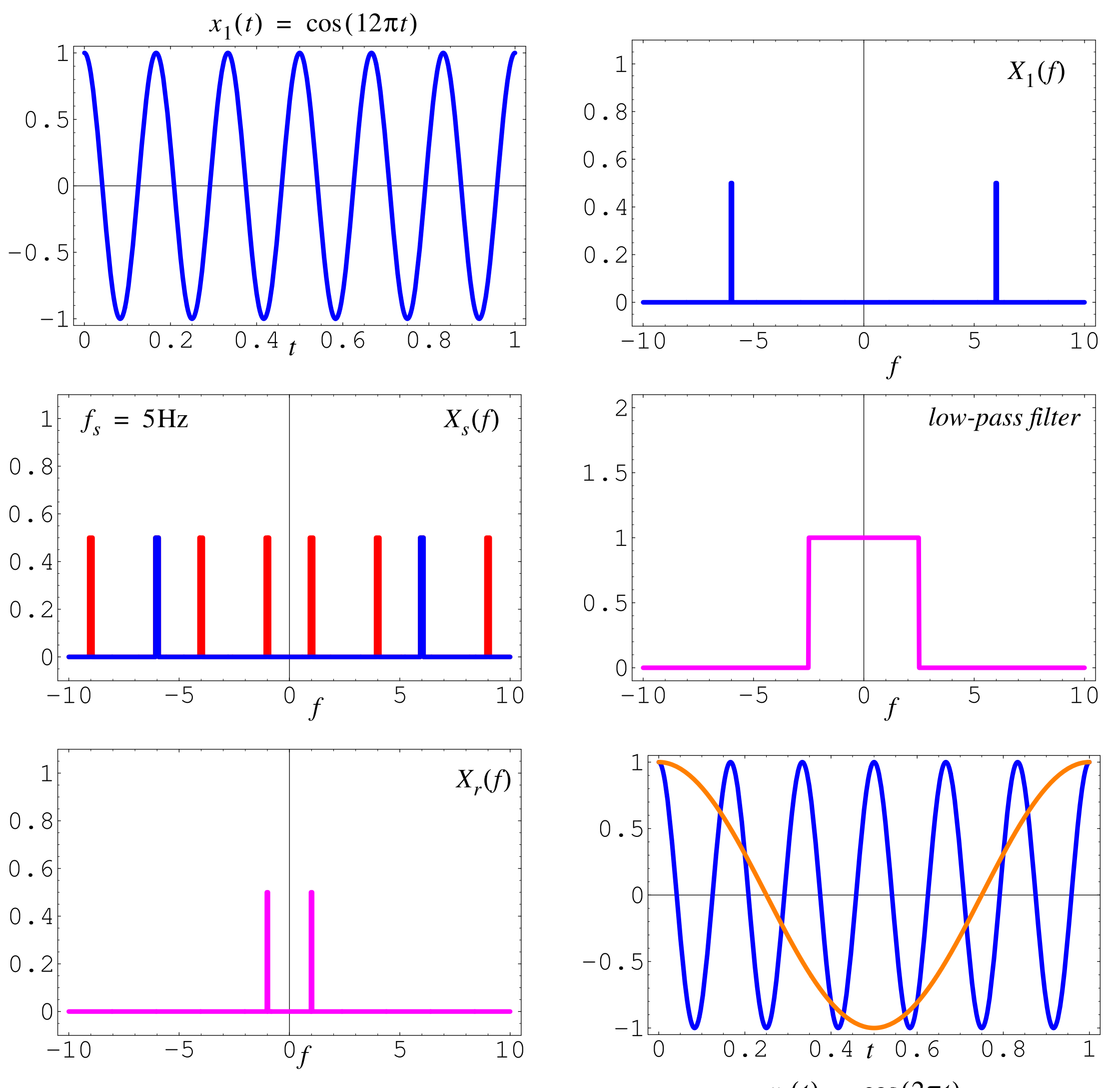


Figure 1

The spectrum $X_s(f)$ of the discrete-time sequence $x[n]$ in the continuous-time domain is given by,

$$X_s(f) = \sum_{k=-\infty}^{\infty} X_1(f - kf_s) \tag{S-2}$$

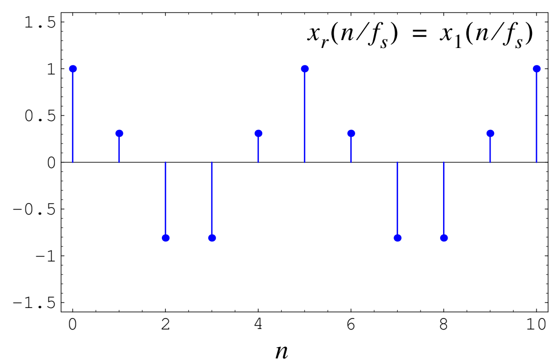
The low-pass filter cuts off all frequencies outside of $\pm f_s/2 = \pm 2.5$ Hz leaving the following frequency components:

$$X_r(f) = \frac{1}{2}\delta(f+1) + \frac{1}{2}\delta(f-1) \tag{S-3}$$

Therefore, the reconstructed waveform $x_r(t)$ is given by,

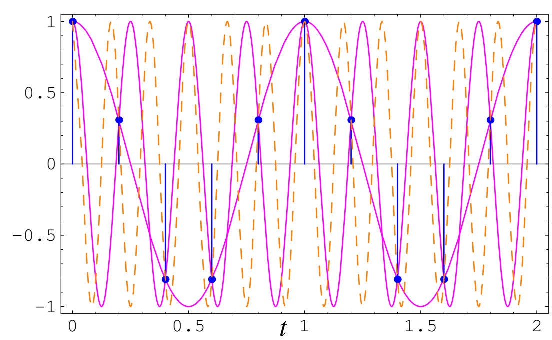
$$x_r(t) = \cos(2\pi t). \tag{S-4}$$

(b) Plot $x_r(n/f_s)$ and $x_1(n/f_s)$ for $n \in \{0, 1, \ldots, 9, 10\}$.



(c) Give analytic expressions for at least two aliases of $x_1(t)$ — that is, continuous-time signals not equal to $x_1(t)$ that will yield the same discrete-time sequences as $x_1[n]$ for $f_s = 5$ Hz.

One alias is $x_r(t) = \cos(2\pi t)$ since it produces an equivalent discrete-time sequence as $x_1(t)$ for $f_s = 5$ Hz. Other aliases (for the sam reason) include $\cos(8\pi t)$, $\cos(18\pi t)$, $\cos(32\pi t)$. The figure below plots $x_1(t)$ (dashed orange line), and two of the above aliases ($\cos(2\pi t)$, $\cos(8\pi t)$ — pink lines); as can be observed, all three functions yield the same discrete time sequence for $f_s = 5$ Hz.



(d) What is the Nyquist sampling frequency corresponding to signal $x_1(t)$?

The Nyquist sampling frequency is given by,

$$f_{Nyquist} = 2\left(\frac{12\pi}{2\pi}\right) = 12 \text{ Hz}. \tag{S-5}$$

Problem 2:

Consider the continuous-time signal:

$$x_2(t) = 1 + 2\cos(2\pi t) + 3\cos(4\pi t). \tag{3}$$

Assume that you sample $x_2(t)$ such that,

$$x_2[n] = x_2(n/f_s) \tag{4}$$

where $f_s = 3$ Hz, and then pass the sampled sequence $x_2[n]$ through an ideal low-pass filter with cut-off frequencies at $\pm f_s/2 = \pm 1.5$ Hz to produce the signal $x_r(t)$.

(a) Give an analytic expression for $x_r(t)$. Be sure to explain all the steps required to get your answer.

Figure 2 illustrates the steps in deriving $x_r(t)$. The spectrum for $x_2(t)$ is given by:

$$X_2(f) = \delta(f) + \delta(f+1) + \delta(f-1) + (3/2)\delta(f+2) + 2\delta(f-2) \tag{S-6}$$

The spectrum $X_s(f)$ of the discrete-time sequence $x[n]$ in the continuous-time domain is given by,

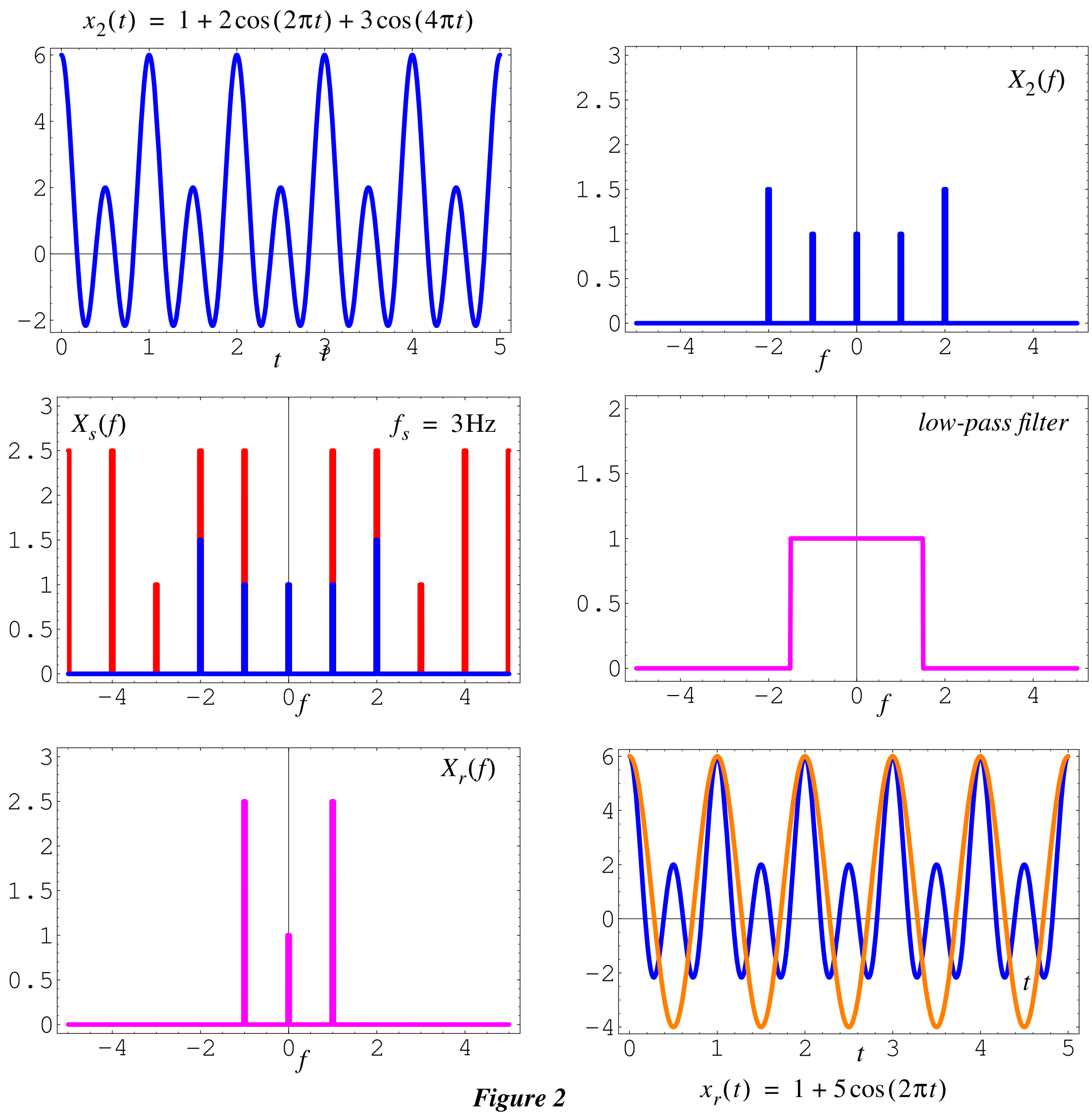


Figure 2

$$X_s(f) = \sum_{k=-\infty}^{\infty} X_2(f - kf_s) \quad \text{(S-7)}$$

The low-pass filter cuts off all frequencies outside of $\pm f_s/2 = \pm 1.5$ Hz leaving the following frequency components:

$$X_r(f) = \delta(f) + (5/2)\delta(f+1) + (5/2)\delta(f-1) \quad \text{(S-8)}$$

Therefore, the reconstructed waveform $x_r(t)$ is given by,

$$x_r(t) = 1 + 5\cos(2\pi t)\,. \quad \text{(S-9)}$$

Note that when frequency components in the sum of equation (S-7) overlap, they add up to create larger frequency components. Also, note that the reconstructed waveform overlaps the original waveform at exactly 1/3 second intervals; that is, where the sampling rate of 3Hz would sample $x_7(t)$.

(b) What is the Nyquist sampling frequency corresponding to signal $x_2(t)$?

The Nyquist sampling frequency is given by,

$$f_{Nyquist} = 2\left(\frac{4\pi}{2\pi}\right) = 4 \text{ Hz.} \quad \text{(S-10)}$$

Problem 3:

Let the frequency representation $X_3(f)$ of a continuous-time signal $x_3(t)$ be given by,

$$X_3(f) = |f|\,[u(f+f_{max}) - u(f-f_{max})]\,. \quad (5)$$

Assume that you sample $x_3(t)$ such that,

$$x_3[n] = x_3(n/f_s)\,, \quad (6)$$

and then pass the sampled sequence $x_3[n]$ through an ideal low-pass filter with cut-off frequencies at $\pm f_s/2$ to produce the signal $x_r(t)$.

(a) Plot the frequency representation $X_r(f)$ of the signal $x_r(t)$ for the sampling frequency $f_s = 3f_{max}$, and give an analytic expression for $X_r(f)$.

Since $f_s > 2f_{max}$, $X_r(f) = X_3(f)$. See Figure 3 for an illustration, where $f_{max} = 1$ Hz.

(b) Plot the frequency representation $X_r(f)$ of the signal $x_r(t)$ for the sampling frequency $f_s = 2f_{max}/3$, and give an analytic expression for $X_r(f)$.

Since $f_s < 2f_{max}$,

$$X_r(f) = \left[\sum_{k=-\infty}^{\infty} X_3(f - kf_s)\right]\left[u\left(f + \frac{f_s}{2}\right) - u\left(f - \frac{f_s}{2}\right)\right] \quad \text{(S-11)}$$

$$X_r(f) = \left[\sum_{k=-\infty}^{\infty} X_3\left(f - \frac{2k}{3}f_{max}\right)\right]\left[u\left(f + \frac{f_{max}}{3}\right) - u\left(f - \frac{f_{max}}{3}\right)\right] \quad \text{(S-12)}$$

As a practical matter we do not have to do an infinite sum in (S-12), since the filter cuts off all frequencies outside the interval $-f_{max}/3 < f < f_{max}/3$:

$$X_r(f) = \left[\sum_{k=-1}^{1} X_3\left(f - \frac{2k}{3}f_{max}\right)\right]\left[u\left(f + \frac{f_{max}}{3}\right) - u\left(f - \frac{f_{max}}{3}\right)\right] \quad \text{(S-13)}$$

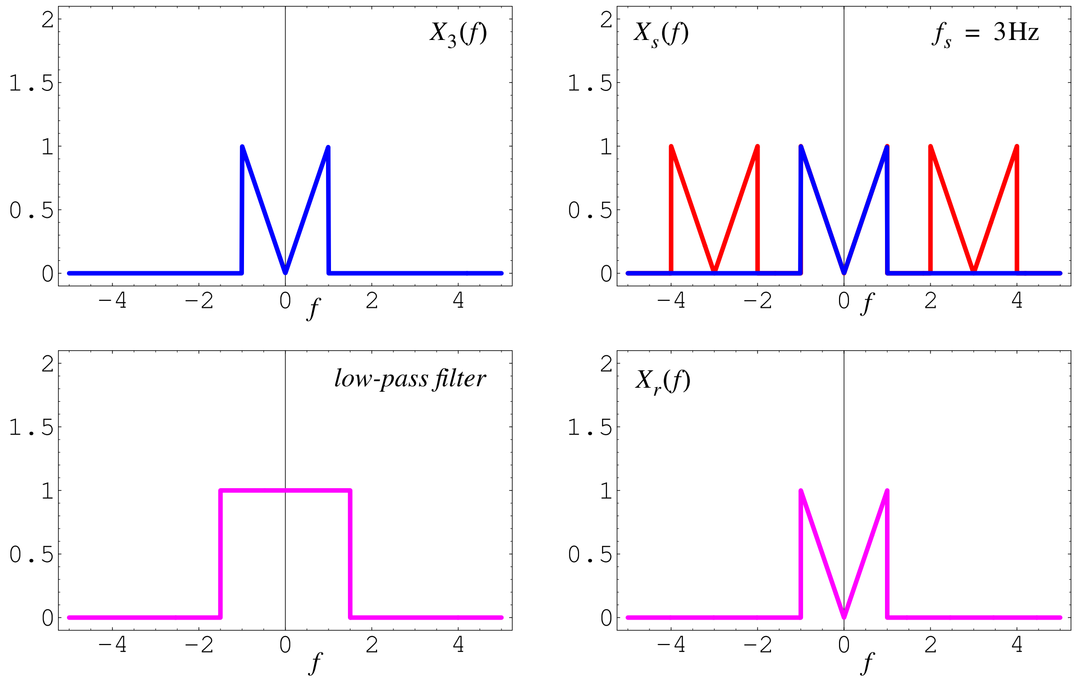


Figure 3

See Figure 4 below for an illustration, where $f_{max} = 1$ Hz.

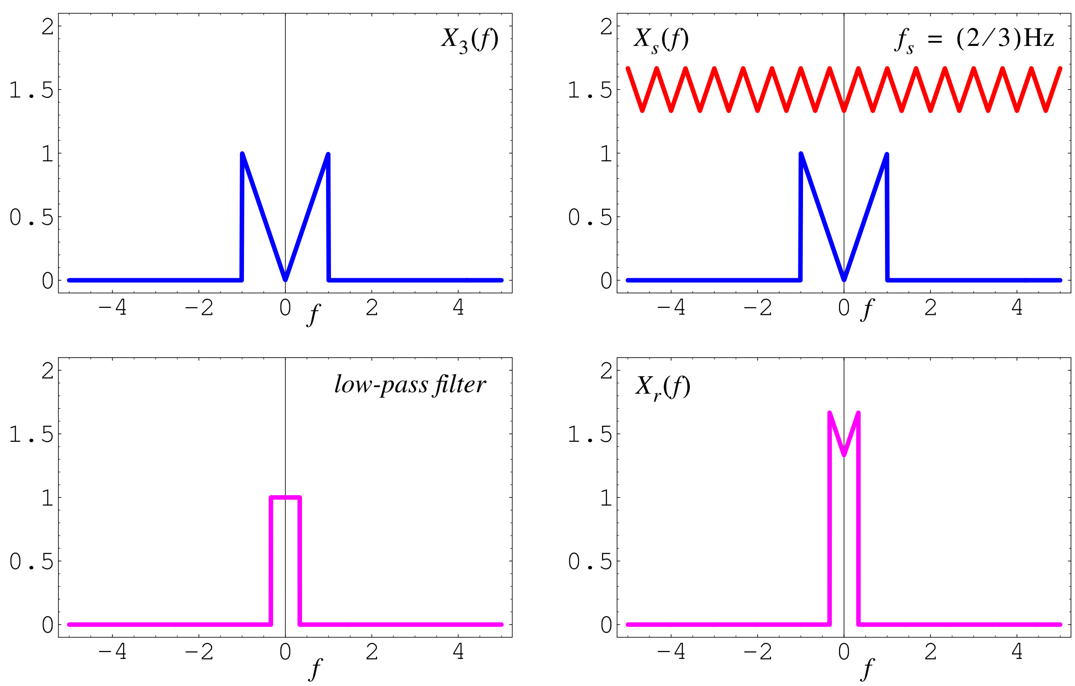


Figure 4

Problem 4:

Assume that you sampled a piece of music at sampling frequency $f_s > 2f_{max}$ but encoded an incorrect sampling frequency in the resulting digital music file (e.g. wav or mp3).

(a) Explain qualitatively what the music would sound like when played back, if the incorrectly encoded sampling frequency f_s' is given by,

$$f_s' = (3f_s)/4. \tag{7}$$

The music will sound slow and muffled, since all the frequencies in the original piece of music will be played at three-fourth their original frequency.

(b) Repeat part (a) for $f_s' = (4f_s)/3$.

The music will sound fast and high-pitched, since all the frequencies in the original piece of music will be played at four-thirds their original frequency.

Problem 5:

Compute $\arg(z)$ for the following complex numbers:

(a) $z = 1 + \mathbf{j}$

(b) $z = 1 - \mathbf{j}$

(c) $-1 - \mathbf{j}$

(d) $-1 + \mathbf{j}$

For part (a), z lies in quadrant I; therefore $\arg(z) = \text{atan}(1/1) = \pi/4$.

For part (b), z lies in quadrant IV; therefore $\arg(z) = \text{atan}(-1/1) = -\pi/4$.

For part (c), z lies in quadrant III; therefore $\arg(z) = -\pi + \text{atan}(-1/-1) = -3\pi/4$.

For part (d), z lies in quadrant II; therefore $\arg(z) = \pi + \text{atan}(1/-1) = 3\pi/4$.

Problem 6:

Let $z_1 = (a + \mathbf{j}b)^2 e^{\mathbf{j}c}$, where a, b and c are real numbers. Solve the following expressions in terms of a, b and c. Your answers should not include the imaginary number $\mathbf{j}$.

(a) $Re[z_1]$

(b) $Im[z_1]$

(c) $Im[\mathbf{j}z_1]$

(d) $|z_1|$

(e) $\arg(z_1)$

(f) $Re[e^{\mathbf{j}\pi/4} z_1]$

Let us first expand z_1:

$$z_1 = (a^2 + \mathbf{j}2ab - b^2)(\cos(c) + \mathbf{j}\sin(c)) \tag{S-14}$$

$$z_1 = (a^2 - b^2)\cos(c) - 2ab\sin(c) + \mathbf{j}(2ab\cos(c) + (a^2 - b^2)\sin(c)) \tag{S-15}$$

Therefore:

$$Re[z_1] = (a^2 - b^2)\cos(c) - 2ab\sin(c) \text{ [part (a)]} \tag{S-16}$$

$$Im[z_1] = 2ab\cos(c) + (a^2 - b^2)\sin(c) \text{ [part (b)]} \tag{S-17}$$

$$Im[\mathbf{j}z_1] = Re[z_1] = (a^2 - b^2)\cos(c) - 2ab\sin(c) \text{ [part (c)]} \tag{S-18}$$

For part (d), we know that $|e^{\mathbf{j}c}| = 1$, so that $|z_1| = |(a + \mathbf{j}b)^2|$:

$$\begin{aligned} |z_1| &= |(a+\mathbf{j}b)^2| = |(a^2+\mathbf{j}2ab-b^2)| \\ &= \sqrt{(a^2-b^2)^2+(2ab)^2} \\ &= \sqrt{a^4-2a^2b^2+b^4+4a^2b^2} \\ &= \sqrt{a^4+2a^2b^2+b^4} \\ &= \sqrt{(a^2+b^2)^2} = (a^2+b^2) \end{aligned} \tag{S-19}$$

$$|z_1| = (a^2+b^2) \text{ [part (d)]} \tag{S-20}$$

For part (e), we can rewrite the first part of z_1 as:

$$(a+\mathbf{j}b)^2 = |(a+\mathbf{j}b)^2| e^{\mathbf{j}\arg(a+\mathbf{j}b)} e^{\mathbf{j}\arg(a+\mathbf{j}b)} = (a^2+b^2)e^{\mathbf{j}2\,\mathrm{atan}(b,\,a)} \tag{S-21}$$

so that:

$$z_1 = (a^2+b^2)e^{\mathbf{j}2\,\mathrm{atan}(b,\,a)}e^{\mathbf{j}c} = (a^2+b^2)e^{\mathbf{j}[2\,\mathrm{atan}(b,\,a)+c]} \tag{S-22}$$

Therefore,

$$\arg(z_1) = 2\,\mathrm{atan}(b,a)+c \text{ [part (e)]} \tag{S-23}$$

Finally, for part (f), we multiply equation (S-22) by $e^{\mathbf{j}\pi/4}$:

$$e^{\mathbf{j}\pi/4}z_1 = (a^2+b^2)e^{\mathbf{j}[2\,\mathrm{atan}(b,\,a)+c]}e^{\mathbf{j}\pi/4} \tag{S-24}$$

$$e^{\mathbf{j}\pi/4}z_1 = (a^2+b^2)e^{\mathbf{j}[2\,\mathrm{atan}(b,\,a)+c+\pi/4]} \tag{S-25}$$

Therefore,

$$Re[e^{\mathbf{j}\pi/4}z_1] = (a^2+b^2)\cos[2\,\mathrm{atan}(b,a)+c+\pi/4] \text{ [part (f)]} \tag{S-26}$$

Problem 7:

Let,

$$z_1 = -2+\mathbf{j}2\sqrt{3} \text{ and } z_2 = 5e^{-\mathbf{j}\pi/3}$$

Solve the following expressions, giving your answers in both polar and rectangular form. You may verify your answers by computer, but must show all work required to arrive at the answers by hand calculation. Your answers should be exact (no numerical approximations).

(a) $2z_1+z_2$

(b) $(z_1z_2)^2$

(c) $z_1z_1{}^*$

(d) z_1/z_2

(e) e^{z_2}

(f) $\sqrt{z_1}$

First, let us write z_1 in polar form:

$$z_1 = \sqrt{(-2)^2+(2\sqrt{3})^2}e^{\mathbf{j}\,\mathrm{atan}(2\sqrt{3},\,-2)} = 4e^{\mathbf{j}2\pi/3} \tag{S-27}$$

and z_2 in rectangular form:

$$z_2 = 5\cos(-\pi/3)+\mathbf{j}5\sin(-\pi/3) = 5/2-\mathbf{j}5\sqrt{3}/2 \tag{S-28}$$

Part (a):

$$2z_1+z_2 = 2(-2+\mathbf{j}2\sqrt{3})+(5/2-\mathbf{j}5\sqrt{3}/2) = -3/2+\mathbf{j}3\sqrt{3}/2 \tag{S-29}$$

In polar form:

$$\begin{aligned} -3/2 + \mathbf{j}3\sqrt{3}/2 &= \sqrt{(-3/2)^2 + (3\sqrt{3}/2)^2}\, e^{\mathbf{j}\,\mathrm{atan}(3\sqrt{3}/2,\,-3/2)} \\ &= \sqrt{36/4}\, e^{\mathbf{j}\,\mathrm{atan}(\sqrt{3},\,-1)} = 3e^{\mathbf{j}2\pi/3} \end{aligned} \tag{S-30}$$

Part (b):

$$(z_1 z_2)^2 = (4e^{\mathbf{j}2\pi/3} \cdot 5e^{-\mathbf{j}\pi/3})^2 = (20e^{\mathbf{j}\pi/3})^2 = 400e^{\mathbf{j}2\pi/3} \tag{S-31}$$

In rectangular form:

$$\begin{aligned} 400e^{\mathbf{j}2\pi/3} &= 400\cos(2\pi/3) + \mathbf{j}400\sin(2\pi/3) \\ &= -200 + \mathbf{j}200\sqrt{3} \end{aligned} \tag{S-32}$$

Part (c):

$$z_1 z_1{}^* = 4e^{\mathbf{j}2\pi/3} \cdot 4e^{-\mathbf{j}2\pi/3} = 16 \tag{S-33}$$

Part (d):

$$z_1/z_2 = (4e^{\mathbf{j}2\pi/3})/(5e^{-\mathbf{j}\pi/3}) = (4/5)e^{\mathbf{j}\pi} = -4/5 \tag{S-34}$$

Part (e):

$$e^{z_2} = \exp[5e^{-\mathbf{j}\pi/3}] = \exp[5/2 - \mathbf{j}5\sqrt{3}/2] = e^{(5/2)}e^{-\mathbf{j}5\sqrt{3}/2} \tag{S-35}$$

In rectangular form:

$$\begin{aligned} e^{(5/2)}e^{-\mathbf{j}5\sqrt{3}/2} &= e^{(5/2)}\cos(-5\sqrt{3}/2) + \mathbf{j}e^{(5/2)}\sin(-5\sqrt{3}/2) \\ &= e^{(5/2)}\cos(5\sqrt{3}/2) - \mathbf{j}e^{(5/2)}\sin(5\sqrt{3}/2) \end{aligned} \tag{S-36}$$

Part (f):

$$\sqrt{z_1} = (4e^{\mathbf{j}2\pi/3})^{(1/2)} = 2e^{\mathbf{j}\pi/3} \tag{S-37}$$

In rectangular form:

$$\begin{aligned} 2e^{\mathbf{j}\pi/3} &= 2\cos(\pi/3) + \mathbf{j}2\sin(\pi/3) \\ &= 1 + \mathbf{j}\sqrt{3} \end{aligned} \tag{S-38}$$

Problem 8:

Using the inverse Euler formulas, prove the following trigonometric identities:

(a) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$\sin(\alpha + \beta) = \left(\frac{e^{\mathbf{j}\alpha} - e^{-\mathbf{j}\alpha}}{2\mathbf{j}}\right)\left(\frac{e^{\mathbf{j}\beta} + e^{-\mathbf{j}\beta}}{2}\right) + \left(\frac{e^{\mathbf{j}\alpha} + e^{-\mathbf{j}\alpha}}{2}\right)\left(\frac{e^{\mathbf{j}\beta} - e^{-\mathbf{j}\beta}}{2\mathbf{j}}\right) \tag{S-39}$$

$$\begin{aligned} \sin(\alpha + \beta) = {} & \frac{e^{\mathbf{j}(\alpha+\beta)} + e^{\mathbf{j}(\alpha-\beta)} - e^{\mathbf{j}(-\alpha+\beta)} - e^{\mathbf{j}(-\alpha-\beta)}}{4\mathbf{j}} + \\ & \frac{e^{\mathbf{j}(\alpha+\beta)} - e^{\mathbf{j}(\alpha-\beta)} + e^{\mathbf{j}(-\alpha+\beta)} - e^{\mathbf{j}(-\alpha-\beta)}}{4\mathbf{j}} \end{aligned} \tag{S-40}$$

$$\sin(\alpha+\beta) = \frac{e^{j(\alpha+\beta)} - e^{j(-\alpha-\beta)}}{4j} + \frac{e^{j(\alpha+\beta)} - e^{j(-\alpha-\beta)}}{4j} + \frac{e^{j(\alpha-\beta)} - e^{j(\alpha-\beta)} + e^{j(-\alpha+\beta)} - e^{j(-\alpha+\beta)}}{4j} \quad \text{(S-41)}$$

$$\sin(\alpha+\beta) = \frac{e^{j(\alpha+\beta)} - e^{-j(\alpha+\beta)}}{2j} \text{ (inverse Euler formula)} \quad \text{(S-42)}$$

(b) $\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$

$$\cos(3\theta) = \left(\frac{e^{j\theta}+e^{-j\theta}}{2}\right)^3 - 3\left(\frac{e^{j\theta}+e^{-j\theta}}{2}\right)\left(\frac{e^{j\theta}-e^{-j\theta}}{2j}\right)^2 \quad \text{(S-43)}$$

Expanding the power subexpressions in (S-43) above:

$$(e^{j\theta}+e^{-j\theta})^3 = (e^{j2\theta}+e^{-j2\theta}+2)(e^{j\theta}+e^{-j\theta}) = e^{j3\theta}+3e^{j\theta}+3e^{-j\theta}+e^{-j3\theta} \quad \text{(S-44)}$$

$$(e^{j\theta}-e^{-j\theta})^2 = e^{j2\theta}+e^{-j2\theta}-2 \quad \text{(S-45)}$$

(S-46)

$$\cos(3\theta) = \frac{1}{8}(e^{j3\theta}+3e^{j\theta}+3e^{-j\theta}+e^{-j3\theta}) + \frac{3}{4}\left(\frac{e^{j\theta}+e^{-j\theta}}{2}\right)(e^{j2\theta}+e^{-j2\theta}-2) \quad \text{(S-47)}$$

$$\cos(3\theta) = \frac{1}{8}(e^{j3\theta}+3e^{j\theta}+3e^{-j\theta}+e^{-j3\theta}) + \frac{3}{8}(e^{j\theta}+e^{-j\theta})(e^{j2\theta}+e^{-j2\theta}-2) \quad \text{(S-48)}$$

$$\cos(3\theta) = \frac{1}{8}(e^{j3\theta}+3e^{j\theta}+3e^{-j\theta}+e^{-j3\theta}) + \frac{3}{8}(e^{j3\theta}+e^{-j\theta}-2e^{j\theta}+e^{j\theta}+e^{-j3\theta}-2e^{-j\theta}) \quad \text{(S-49)}$$

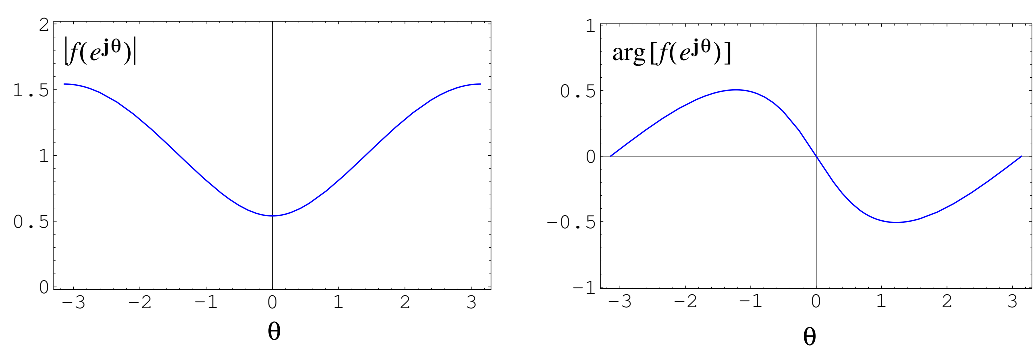
$$\cos(3\theta) = \frac{1}{8}(e^{j3\theta}+3e^{j\theta}+3e^{-j\theta}+e^{-j3\theta}) + \frac{3}{8}(e^{j3\theta}-e^{j\theta}-e^{-j\theta}+e^{-j3\theta}) \quad \text{(S-50)}$$

$$\cos(3\theta) = \frac{4}{8}(e^{j3\theta}+e^{-j3\theta}) \quad \text{(S-51)}$$

$$\cos(3\theta) = \frac{e^{j3\theta}+e^{-j3\theta}}{2} \text{ (inverse Euler formula)} \quad \text{(S-52)}$$

Problem 9:

Let $f(z) = \cos(\sqrt{z})$. Plot $|f(e^{j\theta})|$ and $\arg[f(e^{j\theta})]$ for $-\pi \le \theta < \pi$.



(See the recent class e-mail for the *Mathematica* and *Matlab* code for these plots.)