EEL3135: Homework #4 Solutions

Problem 1:

For each of the systems below, determine whether or not the system is (1) linear, (2) time-invariant, and (3) causal:

(a) $y[n] = x[n]\cos(0.4\pi n)$

(b) $y[n] = x[n] - x[n-5]$

(c) $y[n] = x[|n|]$

(d) $y[n] = |x[n]|$

(e) $y[n] = 2x[n-2] + x[n+1]$

(f) $y[n] = 3x[n-2] + \sqrt{n}$

Solution: Linearity: For each difference equation above, we compute $w[n]$ and $y[n]$ in Figure 1 below; if the two outcomes are equal, the system is linear; if not, the system is not linear.

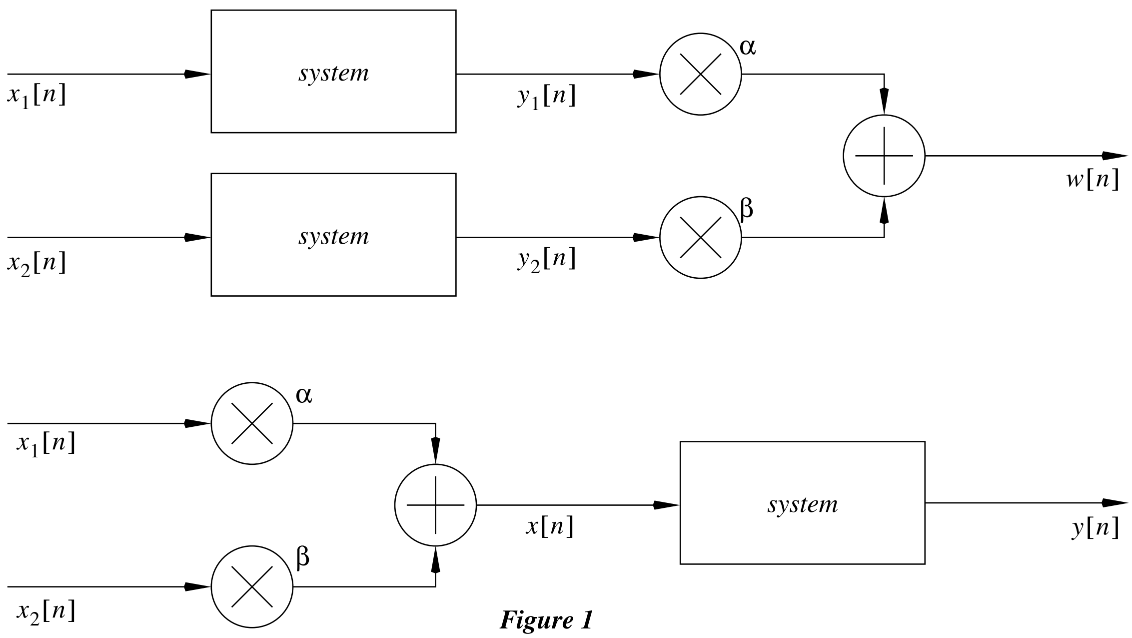

Figure 1

Part (a):

$$y_1[n] = x_1[n]\cos(0.4\pi n),\ y_2[n] = x_2[n]\cos(0.4\pi n) \quad \text{(S-1)}$$

$$w[n] = \alpha y_1[n] + \beta y_2[n] = \alpha x_1[n]\cos(0.4\pi n) + \beta x_2[n]\cos(0.4\pi n) \quad \text{(S-2)}$$

Next:

$$x[n] = \alpha x_1[n] + \beta x_2[n] \quad \text{(S-3)}$$

$$y[n] = x[n]\cos(0.4\pi n) = (\alpha x_1[n] + \beta x_2[n])\cos(0.4\pi n) \quad \text{(S-4)}$$

$$y[n] = \alpha x_1[n]\cos(0.4\pi n) + \beta x_2[n]\cos(0.4\pi n) \quad \text{(S-5)}$$

Since the results in equations (S-2) and (S-5) are the same, system (a) is *linear.*

Part (b):

$$y_1[n] = x_1[n] - x_1[n-5],\ y_2[n] = x_2[n] - x_2[n-5] \quad \text{(S-6)}$$

$$w[n] = \alpha y_1[n] + \beta y_2[n] = \alpha(x_1[n] - x_1[n-5]) + \beta(x_2[n] - x_2[n-5]) \quad \text{(S-7)}$$

Next:

$$x[n] = \alpha x_1[n] + \beta x_2[n] \quad \text{(S-8)}$$

$$y[n] = x[n] - x[n-5] = (\alpha x_1[n] + \beta x_2[n]) - (\alpha x_1[n-5] + \beta x_2[n-5]) \quad \text{(S-9)}$$

$$y[n] = \alpha(x_1[n] - x_1[n-5]) + \beta(x_2[n] - x_2[n-5]) \quad \text{(S-10)}$$

Since the results in equations (S-7) and (S-10) are the same, system (b) is *linear.*

Part (c):

$$y_1[n] = x_1[|n|],\ y_2[n] = x_2[|n|] \quad \text{(S-11)}$$

$$w[n] = \alpha y_1[n] + \beta y_2[n] = \alpha x_1[|n|] + \beta x_2[|n|] \quad \text{(S-12)}$$

Next:

$$x[n] = \alpha x_1[n] + \beta x_2[n] \quad \text{(S-13)}$$

$$y[n] = x[|n|] = \alpha x_1[|n|] + \beta x_2[|n|] \quad \text{(S-14)}$$

Since the results in equations (S-12) and (S-14) are the same, system (c) is *linear.*

Part (d):

$$y_1[n] = |x_1[n]|,\ y_2[n] = |x_2[n]| \quad \text{(S-15)}$$

$$w[n] = \alpha y_1[n] + \beta y_2[n] = \alpha|x_1[n]| + \beta|x_2[n]| \quad \text{(S-16)}$$

Next:

$$x[n] = \alpha x_1[n] + \beta x_2[n] \quad \text{(S-17)}$$

$$y[n] = |x[n]| = |\alpha x_1[n] + \beta x_2[n]| \quad \text{(S-18)}$$

Since the results in equations (S-16) and (S-18) are not equal, system (d) is *nonlinear.*

Part (e):

$$y_1[n] = 2x_1[n-2] + x_1[n+1],\ y_2[n] = 2x_2[n-2] + x_2[n+1] \quad \text{(S-19)}$$

$$w[n] = \alpha y_1[n] + \beta y_2[n] = \alpha(2x_1[n-2] + x_1[n+1]) + \beta(2x_2[n-2] + x_2[n+1]) \quad \text{(S-20)}$$

Next:

$$x[n] = \alpha x_1[n] + \beta x_2[n] \quad \text{(S-21)}$$

$$y[n] = 2x[n-2] + x[n+1] = 2(\alpha x_1[n-2] + \beta x_2[n-2]) + \alpha x_1[n+1] + \beta x_2[n+1] \quad \text{(S-22)}$$

$$y[n] = \alpha(2x_1[n-2] + x_1[n+1]) + \beta(2x_2[n-2] + x_2[n+1]) \quad \text{(S-23)}$$

Since the results in equations (S-20) and (S-23) are the same, system (e) is *linear.*

Part (f):

$$y_1[n] = 3x_1[n-2] + \sqrt{n},\ y_2[n] = 3x_2[n-2] + \sqrt{n} \quad \text{(S-24)}$$

$$w[n] = \alpha y_1[n] + \beta y_2[n] = \alpha(3x_1[n-2] + \sqrt{n}) + \beta(3x_2[n-2] + \sqrt{n}) \quad \text{(S-25)}$$

$$w[n] = 3(\alpha + \beta)x_1[n-2] + (\alpha + \beta)\sqrt{n} \quad \text{(S-26)}$$

Next:

$$x[n] = \alpha x_1[n] + \beta x_2[n] \quad \text{(S-27)}$$

$$y[n] = 3x[n-2] + \sqrt{n} = 3(\alpha x_1[n-2] + \beta x_2[n-2]) + \sqrt{n} \quad \text{(S-28)}$$

$$y[n] = 3(\alpha + \beta)x_1[n-2] + \sqrt{n} \quad \text{(S-29)}$$

Since the results in equations (S-26) and (S-29) are not equal, system (f) is *nonlinear*.

Solution: Time-invariance: For each difference equation above, we compute $w[n]$ and $y[n-n_0]$ in Figure 2 below; if the two outcomes are equal, the system is time-invariant; if not, the system is time-variant.

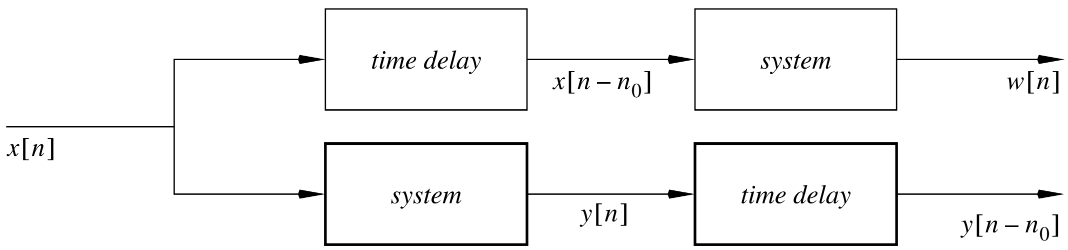

Figure 2

Part (a):

$$w[n] = x[n-n_0]\cos(0.4\pi n) \quad \text{(S-30)}$$

$$y[n-n_0] = x[n-n_0]\cos(0.4\pi(n-n_0)) \quad \text{(S-31)}$$

Since the results in equations (S-30) and (S-31) are not equal, system (a) is *time-variant*.

Part (b):

$$w[n] = x[n-n_0] - x[n-n_0-5] \quad \text{(S-32)}$$

$$y[n-n_0] = x[n-n_0] - x[n-n_0-5] \quad \text{(S-33)}$$

Since the results in equations (S-32) and (S-33) are the same, system (b) is *time-invariant*.

Part (c):

$$w[n] = x_1[|n| - n_0] \quad \text{(S-34)}$$

$$y[n-n_0] = x[|n-n_0|] \quad \text{(S-35)}$$

Since the results in equations (S-34) and (S-35) are not equal, system (c) is *time-variant*.

Part (d):

$$w[n] = |x[n-n_0]|\ ,\ y[n-n_0] = |x[n-n_0]| \quad \text{(S-36)}$$

Since the results in (S-36) are equal, system (d) is *time-invariant*.

Part (e):

$$w[n] = 2x[n-n_0-2]+x[n-n_0+1] \tag{S-37}$$

$$y[n-n_0] = 2x[n-n_0-2]+x[n-n_0+1] \tag{S-38}$$

Since the results in equations (S-37) and (S-38) are the same, system (e) is *time-invariant*.

Part (f):

$$w[n] = 3x[n-n_0-2]+\sqrt{n} \tag{S-39}$$

$$y[n-n_0] = 3x[n-n_0-2]+\sqrt{(n-n_0)} \tag{S-40}$$

Since the results in equations (S-39) and (S-40) are not equal, system (f) is *time-variant*.

<u>Solution</u>: Causality: For each difference equation, we determine whether the system output is zero prior to any input to the system; if so, the system is causal; if not the system is noncausal.

System (a) is *causal* (by inspection).

System (b) is *causal* (by inspection).

System (c) is *noncausal*. Consider, for example an input $x[n] = \delta[n-2]$; the corresponding output is then given by:

$$y[n] = \delta[n+2]+\delta[n-2]\,. \tag{S-41}$$

System (d) is *causal* (by inspection).

System (e) is *noncausal*. Consider, for example an input $x[n] = \delta[n]$; the corresponding output is then given by:

$$h[n] = \delta[n-2]+\delta[n+1]\,. \tag{S-42}$$

System (f) is *noncausal*, because $y[n]$ is nonzero for all n, even with zero input. That is, if $x[n] = 0$,

$$y[n] = \sqrt{n}\,,\ -\infty < n < \infty\,. \tag{S-43}$$

The table below summarizes the solutions for problem 1:

System	*Linear?*	*Time-invariant?*	*Causal?*
$y[n] = x[n]\cos(0.4\pi n)$	yes	no	yes
$y[n] = x[n]-x[n-5]$	yes	yes	yes
$y[n] = x[\lvert n\rvert]$	yes	no	no
$y[n] = \lvert x[n]\rvert$	no	yes	yes
$y[n] = 2x[n-2]+x[n+1]$	yes	yes	no
$y[n] = 3x[n-2]+\sqrt{n}$	no	no	no

Problem 2:

Assume the impulse response of an LTI system is $h[n]$ given below:

$$h[n] = \delta[n+1] + 2\delta[n-2] - 3\delta[n-4] \tag{2-1}$$

(a) Give the difference equation for this LTI system.

(b) Compute the output $y[n]$ for an input of:

$$x[n] = 4\delta[n] + 2\delta[n-1] + 4\delta[n-4] - \delta[n-5] + 6\delta[n-6] \tag{2-2}$$

using the convolution operator for the range of n where $y[n] \neq 0$ (set up a table similar to one of the two tables in the notes).

(c) Verify your answer in part (b) by direct substitution into the difference equation for part (a).

Solution:

Part (a): We start with equation (2-1) and make the following substitutions: $\delta[n] \Leftrightarrow x[n]$ and $h[n] \Leftrightarrow y[n]$. Therefore, the difference equation of the LTI system is given by,

$$y[n] = x[n+1] + 2x[n-2] - 3x[n-4] \tag{S-44}$$

Part (b): The table below illustrates the convolution procedure for this problem, identical to the method discussed in the lecture notes (Table 2). It indicates all nonzero values of $y[n]$.

n	-1	0	1	2	3	4	5	6	7	8	9	10
$x[n]$		4	2	0	0	4	-1	6				
$h[n]$	1	0	0	2	0	-3						
$h[-1]x[n+1]$	4	2	0	0	4	-1	6					
$h[2]x[n-2]$				8	4	0	0	8	-2	12		
$h[4]x[n-4]$						-12	-6	0	0	-12	3	-18
$y[n]$	4	2	0	8	8	-13	0	8	-2	0	3	-18

Part (c): Below we verify our results above by direct substitution into the difference equation (S-44):

$$y[-1] = x[0] + 2x[-3] - 3x[-5] = 4 \tag{S-45}$$

$$y[0] = x[1] + 2x[-2] - 3x[-4] = 2 \tag{S-46}$$

$$y[1] = x[2] + 2x[-1] - 3x[-3] = 0 \tag{S-47}$$

$$y[2] = x[3] + 2x[0] - 3x[-2] = 8 \tag{S-48}$$

$$y[3] = x[4] + 2x[1] - 3x[-1] = 8 \tag{S-49}$$

$$y[4] = x[5] + 2x[2] - 3x[0] = -13 \tag{S-50}$$

$$y[5] = x[6] + 2x[3] - 3x[1] = 0 \tag{S-51}$$

$$y[6] = x[7] + 2x[4] - 3x[2] = 8 \tag{S-52}$$

$$y[7] = x[8] + 2x[5] - 3x[3] = -2 \tag{S-53}$$

$$y[8] = x[9] + 2x[6] - 3x[4] = 0 \tag{S-54}$$

$$y[9] = x[10] + 2x[7] - 3x[5] = 3 \tag{S-55}$$

$$y[10] = x[11] + 2x[8] - 3x[6] = -18 \tag{S-56}$$

Problem 3:

Assume that the response of an LTI system to the input $x[n]$,

$$x[n] = \delta[n] - \delta[n-1] \tag{3-1}$$

is given by,

$$y[n] = \delta[n] - \delta[n-1] + 2\delta[n-3]\,. \tag{3-2}$$

For this system, compute the output $y_2[n]$ to the following input:

$$x_2[n] = 5\delta[n] - 5\delta[n-3]\,. \tag{3-3}$$

Solution: Since we are not given the impulse response of this LTI system, we must try to express $x_2[n]$ as a linear combination of weighted and time-shifted $x[n]$; then the output $y_2[n]$ will be the same linear combination of weighted and time-shifted $y[n]$. First let us write down the following three expressions:

$$x[n] = \delta[n] - \delta[n-1] \tag{S-57}$$

$$x[n-1] = \delta[n-1] - \delta[n-2] \tag{S-58}$$

$$x[n-2] = \delta[n-2] - \delta[n-3] \tag{S-59}$$

Note that if we add equations (S-57) through (S-59) we get closer to $x_2[n]$:

$$x[n] + x[n-1] + x[n-2] = \delta[n] - \delta[n-3] \tag{S-60}$$

Multiplying equation (S-60) by five, we get $x_2[n]$ in terms of $x[n]$:

$$x_2[n] = 5(x[n] + x[n-1] + x[n-2]) = 5x[n] + 5x[n-1] + 5x[n-2]\,. \tag{S-61}$$

Since the system is linear and time-invariant (LTI), the output $y_2[n]$ corresponding to $x_2[n]$ is therefore given by,

$$y_2[n] = 5y[n] + 5y[n-1] + 5y[n-2] \tag{S-62}$$

$$\begin{aligned} y_2[n] = {} & 5(\delta[n] - \delta[n-1] + 2\delta[n-3]) + 5(\delta[n-1] - \delta[n-2] + 2\delta[n-4]) + \\ & 5(\delta[n-2] - \delta[n-3] + 2\delta[n-5]) \end{aligned} \tag{S-63}$$

$$y_2[n] = 5\delta[n] + 5\delta[n-3] + 10\delta[n-4] + 10\delta[n-5]\,. \tag{S-64}$$

Problem 4:

Assume that the response of an LTI system to the input $x[n] = u[n]$ (discrete-time unit step) is given by,

$$y[n] = \delta[n] + 2\delta[n-1] - \delta[n-2]. \quad (4\text{-}1)$$

(a) For this system, compute the output $y_2[n]$ to the following input:

$$x_2[n] = 3u[n] - 2u[n-4] \quad (4\text{-}2)$$

(b) Derive the impulse response $h[n]$ for this system.

(c) Give the difference equation for this system.

Solution:

Part (a): Similar to problem 3, for an LTI system,

$$x_2[n] = 3x[n] - 2x[n-4] \quad (S\text{-}65)$$

implies,

$$y_2[n] = 3y[n] - 2y[n-4]. \quad (S\text{-}66)$$

Simplifying equation (S-66), we get:

$$y_2[n] = 3(\delta[n] + 2\delta[n-1] - \delta[n-2]) - 2(\delta[n-4] + 2\delta[n-5] - \delta[n-6]) \quad (S\text{-}67)$$

$$y_2[n] = 3\delta[n] + 6\delta[n-1] - 3\delta[n-2] - 2\delta[n-4] - 4\delta[n-5] + 2\delta[n-6]. \quad (S\text{-}68)$$

Part (b): Note that:

$$\delta[n] = u[n] - u[n-1] = x[n] - x[n-1] \quad (S\text{-}69)$$

so that $h[n]$ is given by,

$$h[n] = y[n] - y[n-1] \quad (S\text{-}70)$$

$$h[n] = (\delta[n] + 2\delta[n-1] - \delta[n-2]) - (\delta[n-1] + 2\delta[n-2] - \delta[n-3]) \quad (S\text{-}71)$$

$$h[n] = \delta[n] + \delta[n-1] - 3\delta[n-2] + \delta[n-3]. \quad (S\text{-}72)$$

Part (c): We start with equation (S-72) and make the following substitutions:

$$\delta[n] \Leftrightarrow x[n] \quad (S\text{-}73)$$

$$h[n] \Leftrightarrow y[n] \quad (S\text{-}74)$$

Therefore, the difference equation of the system is given by,

$$y[n] = x[n] + x[n-1] - 3x[n-2] + x[n-3]. \quad (S\text{-}75)$$

Problem 5:

Assume an LTI system of the following form:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \tag{5-1}$$

with impulse response,

$$h[n] = 3\delta[n] + 7\delta[n-1] + 13\delta[n-2] + 9\delta[n-3] + 5\delta[n-4] \tag{5-2}$$

(a) Give numeric values for M and b_k, $k \in \{0, \ldots, M\}$.

(b) Compute $y[n]$, $\forall n$, for input $x[n]$ given by,

$$x[n] = \begin{cases} 0 & n = even \\ 1 & n = odd \end{cases} \tag{5-3}$$

Solution:

Part (a): We start with equation (5-2) and make the following substitutions: $\delta[n] \Leftrightarrow x[n]$ and $h[n] \Leftrightarrow y[n]$. Therefore, the difference equation of the system is given by,

$$y[n] = 3x[n] + 7x[n-1] + 13x[n-2] + 9x[n-3] + 5x[n-4] \tag{S-76}$$

Comparing equations (5-1).and (S-76), we get the following values:

$$b_0 = 3,\ b_1 = 7,\ b_2 = 13,\ b_3 = 9,\ b_4 = 5 \text{ and } M = 4. \tag{S-77}$$

Part (b): We can use convolution or direct substitution. Below, I choose direct substitution into difference equation (S-76):

$$\begin{aligned} y[0] &= 3x[0] + 7x[-1] + 13x[-2] + 9x[-3] + 5x[-4] \\ &= 3(0) + 7(1) + 13(0) + 9(1) + 5(0) \\ &= 16 \end{aligned} \tag{S-78}$$

$$\begin{aligned} y[1] &= 3x[1] + 7x[0] + 13x[-1] + 9x[-2] + 5x[-3] \\ &= 3(1) + 7(0) + 13(1) + 9(0) + 5(1) \\ &= 21 \end{aligned} \tag{S-79}$$

Note that since the input $x[n]$ is periodic with discrete-time period two, we need not compute any additional output values, since:

$$y[2n] = y[0] = 16 \text{ and } y[2n+1] = y[1] = 21,\ \forall n. \tag{S-80}$$

In summary:

$$y[n] = \begin{cases} 16 & n = even \\ 21 & n = odd \end{cases}. \tag{S-81}$$

Problem 6:

For each part below, give an expression for $y[n]$. When applicable, assume $x[n] = 0$, $n < 0$; also, $u[n]$ denotes the discrete-time unit step function, and * denotes the convolution operator.

(a) $y[n] = u[n] * u[n]$

Solution: From the definition of convolution:

$$y[n] = \sum_{k=-\infty}^{\infty} u[k]u[n-k] = \sum_{k=0}^{n} u[k]u[n-k] = \sum_{k=0}^{n} 1 = (n+1),\ n \geq 0 \qquad \text{(S-82)}$$

$$y[n] = (n+1)u[n]. \qquad \text{(S-83)}$$

Note that in equation (S-82) above, we could change the limits in the summation to $k \in \{0, 1, \ldots, n\}$, because $u[k] = 0$ for $k < 0$, and $u[n-k] = 0$ for $k > n$.

(b) $y[n] = x[n] * u[n]$

Solution: From the definition of convolution:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]u[n-k] = \sum_{k=0}^{n} x[k]u[n-k] = \sum_{k=0}^{n} x[k],\ n \geq 0 \qquad \text{(S-84)}$$

$$y[n] = \left(\sum_{k=0}^{n} x[k]\right)u[n]. \qquad \text{(S-85)}$$

Note that in equation (S-84) above, we could change the limits in the summation to $k \in \{0, 1, \ldots, n\}$, because $x[k] = 0$ for $k < 0$ (by problem statement), and $u[n-k] = 0$ for $k > n$.

(c) $y[n] = u[n] * x[n] * \delta[n]$

Solution: Let us denote $y'[n] = u[n] * x[n]$. Then,

$$y[n] = y'[n] * \delta[n] = \sum_{k=-\infty}^{\infty} y'[k]\delta[n-k] = y'[n]. \qquad \text{(S-86)}$$

That is, convolution with $\delta[n]$ does not affect the original signal. Applying the commutative property to the answer for part (b):

$$y'[n] = u[n] * x[n] = x[n] * u[n] = \left(\sum_{k=0}^{n} x[k]\right)u[n]. \qquad \text{(S-87)}$$

Combining the results for (S-86) and (S-87):

$$y[n] = y'[n] = \left(\sum_{k=0}^{n} x[k]\right)u[n]. \qquad \text{(S-88)}$$

(d) $y[n] = a^n u[n] * u[n]$, $|a| < 1$.

Solution: From part (a),

$$u[n] * u[n] = (n+1)u[n] \qquad \text{(S-89)}$$

so that,

$$y[n] = a^n(n+1)u[n]. \qquad \text{(S-90)}$$

Problem 7:

(a) Compute the impulse response $h[n]$ for a filter with frequency response $H(e^{\mathbf{j}\theta})$, as illustrated in Figure 3 below, using the inverse DTFT integral. Simplify your answer as much as possible.

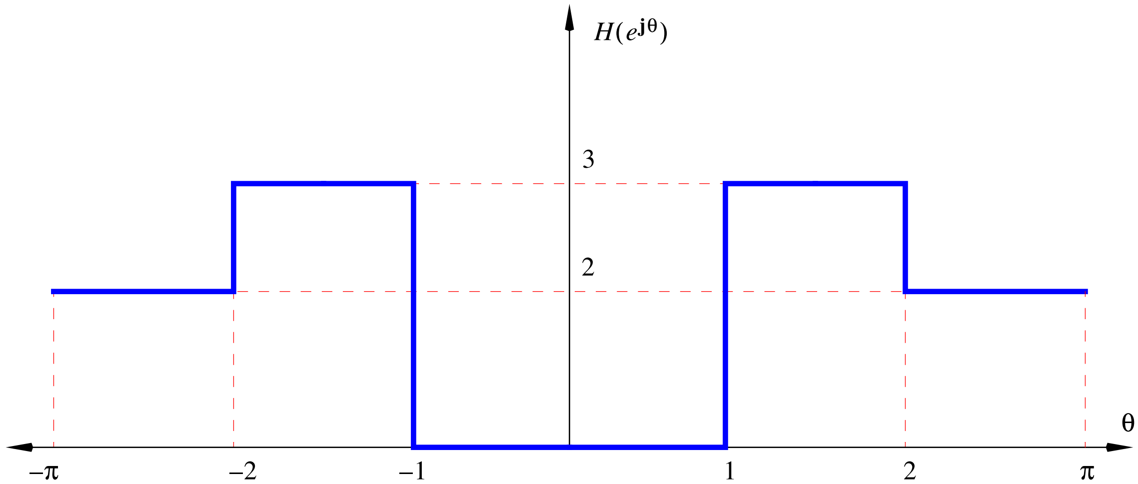

Figure 3

Solution:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\mathbf{j}\theta})e^{\mathbf{j}n\theta}d\theta \tag{S-91}$$

$$h[n] = \frac{1}{2\pi}\left[\int_{-\pi}^{-2} 2e^{\mathbf{j}n\theta}d\theta + \int_{-2}^{-1} 3e^{\mathbf{j}n\theta}d\theta + \int_{1}^{2} 3e^{\mathbf{j}n\theta}d\theta + \int_{2}^{\pi} 2e^{\mathbf{j}n\theta}d\theta\right] \tag{S-92}$$

$$h[n] = \frac{1}{2\pi}\left[\frac{2e^{\mathbf{j}n\theta}}{\mathbf{j}n}\bigg|_{\theta=-\pi}^{\theta=-2} + \frac{3e^{\mathbf{j}n\theta}}{\mathbf{j}n}\bigg|_{\theta=-2}^{\theta=-1} + \frac{3e^{\mathbf{j}n\theta}}{\mathbf{j}n}\bigg|_{\theta=1}^{\theta=2} + \frac{2e^{\mathbf{j}n\theta}}{\mathbf{j}n}\bigg|_{\theta=2}^{\theta=\pi}\right] \tag{S-93}$$

$$h[n] = \frac{1}{2\pi}\left[\left(\frac{2e^{-\mathbf{j}2n}}{\mathbf{j}n} - \frac{2e^{-\mathbf{j}\pi n}}{\mathbf{j}n}\right) + \left(\frac{3e^{-\mathbf{j}n}}{\mathbf{j}n} - \frac{3e^{-\mathbf{j}2n}}{\mathbf{j}n}\right) + \left(\frac{3e^{\mathbf{j}2n}}{\mathbf{j}n} - \frac{3e^{\mathbf{j}n}}{\mathbf{j}n}\right) + \left(\frac{2e^{\mathbf{j}\pi n}}{\mathbf{j}n} - \frac{2e^{\mathbf{j}2n}}{\mathbf{j}n}\right)\right] \tag{S-94}$$

$$h[n] = \frac{1}{2\pi}\left[\left(\frac{e^{\mathbf{j}2n} - e^{-\mathbf{j}2n}}{\mathbf{j}n}\right) - \left(\frac{3e^{\mathbf{j}n} - 3e^{-\mathbf{j}n}}{\mathbf{j}n}\right) + \left(\frac{2e^{\mathbf{j}\pi n} - 2e^{-\mathbf{j}\pi n}}{\mathbf{j}n}\right)\right] \tag{S-95}$$

$$h[n] = \frac{1}{\pi n}\left[\left(\frac{e^{\mathbf{j}2n} - e^{-\mathbf{j}2n}}{\mathbf{j}2}\right) - 3\left(\frac{e^{\mathbf{j}n} - e^{-\mathbf{j}n}}{\mathbf{j}2}\right) + 2\left(\frac{e^{\mathbf{j}\pi n} - e^{-\mathbf{j}\pi n}}{\mathbf{j}2}\right)\right] \tag{S-96}$$

$$h[n] = \frac{1}{\pi n}[\sin(2n) - 3\sin(n) + 2\sin(\pi n)] \tag{S-97}$$

$$h[n] = \frac{\sin(2n) - 3\sin(n)}{\pi n} + 2\delta[n] \tag{S-98}$$

(b) Using L'Hopital's rule, simplify the formula for a general filter $h_g[n]$, derived in the notes, for $\Delta \to 0$, where,

$$h_g[n] = (1-a)\left[\frac{\cos(n(\theta_{c_1} - \Delta)) + \cos(n(\theta_{c_2} + \Delta)) - \cos(n\theta_{c_1}) - \cos(n\theta_{c_2})}{n^2\pi\Delta}\right] + \delta[n] \tag{7-1}$$

Solution:

$$\lim_{\Delta \to 0} h_g[n] = \lim_{\Delta \to 0} (1-a)\left[\frac{\frac{d}{d\Delta}\{\cos(n(\theta_{c_1}-\Delta)) + \cos(n(\theta_{c_2}+\Delta)) - \cos(n\theta_{c_1}) - \cos(n\theta_{c_2})\}}{\frac{d}{d\Delta}\{n^2\pi\Delta\}}\right] + \delta[n] \quad \text{(S-99)}$$

$$\lim_{\Delta \to 0} h_g[n] = \lim_{\Delta \to 0} (1-a)\left[\frac{n\sin(n(\theta_{c_1}-\Delta)) - n\sin(n(\theta_{c_2}+\Delta))}{n^2\pi}\right] + \delta[n] \quad \text{(S-100)}$$

$$\lim_{\Delta \to 0} h_g[n] = (1-a)\left[\frac{\sin(n\theta_{c_1}) - \sin(n\theta_{c_2})}{n\pi}\right] + \delta[n] \quad \text{(S-101)}$$

(c) Explain how you can use your result in part (b) to derive the impulse response $h[n]$ for part (a), without explicit computation of the inverse DTFT integral.

Solution: We can define $H(e^{j\theta})$ as the composition of three filters with the following values:

$$H_1(e^{j\theta}) = H_g(e^{j\theta})\Big|_{a=0,\ \theta_{c_1}=0,\ \theta_{c_2}=1,\ \Delta=0} \quad \text{(S-102)}$$

$$H_2(e^{j\theta}) = H_g(e^{j\theta})\Big|_{a=3,\ \theta_{c_1}=1,\ \theta_{c_2}=2,\ \Delta=0} \quad \text{(S-103)}$$

$$H_3(e^{j\theta}) = H_g(e^{j\theta})\Big|_{a=2,\ \theta_{c_1}=2,\ \theta_{c_2}=\pi,\ \Delta=0} \quad \text{(S-104)}$$

$$H(e^{j\theta}) = H_1(e^{j\theta})H_2(e^{j\theta})H_3(e^{j\theta}) \quad \text{(S-105)}$$

where $H_g(e^{j\theta})$ denotes the DTFT of $h_g[n]$. In the discrete-time domain, equation (S-105) is given by,

$$h[n] = h_1[n] * h_2[n] * h_3[n]. \quad \text{(S-106)}$$

Note that the impulse responses in equation (S-106) are given by,

$$h_1[n] = \lim_{\Delta \to 0} h_g[n] = (1-a)\left[\frac{\sin(n\theta_{c_1}) - \sin(n\theta_{c_2})}{n\pi}\right] + \delta[n] \text{ for } a = 0, \theta_{c_1} = 0, \theta_{c_2} = 1 \quad \text{(S-107)}$$

$$h_1[n] = \frac{-\sin(n)}{n\pi} + \delta[n] \quad \text{(S-108)}$$

$$h_2[n] = \lim_{\Delta \to 0} h_g[n] = (1-a)\left[\frac{\sin(n\theta_{c_1}) - \sin(n\theta_{c_2})}{n\pi}\right] + \delta[n] \text{ for } a = 3, \theta_{c_1} = 1, \theta_{c_2} = 2 \quad \text{(S-109)}$$

$$h_2[n] = 2\left[\frac{\sin(2n) - \sin(n)}{n\pi}\right] + \delta[n] \quad \text{(S-110)}$$

$$h_3[n] = \lim_{\Delta \to 0} h_g[n] = (1-a)\left[\frac{\sin(n\theta_{c_1}) - \sin(n\theta_{c_2})}{n\pi}\right] + \delta[n] \text{ for } a = 2, \theta_{c_1} = 2, \theta_{c_2} = \pi \quad \text{(S-111)}$$

$$h_3[n] = \left[\frac{\sin(n\pi) - \sin(2n)}{n\pi}\right] + \delta[n] \quad \text{(S-112)}$$

Problem 8:

Given an IIR filter defined by the difference equation:

$$y[n] = \left(-\frac{1}{2}\right)y[n-1] + x[n] \quad (8\text{-}1)$$

(a) Determine the transfer function $H(z)$ for this system.

Solution:

$$Y(z) = (-1/2)z^{-1}Y(z) + X(z) \quad (\text{S-113})$$

$$(1 + (1/2)z^{-1})Y(z) = X(z) \quad (\text{S-114})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + (1/2)z^{-1}} \quad (\text{S-115})$$

(b) Compute the system poles.

Solution:

$$1 + (1/2)z^{-1} = 0 \quad (\text{S-116})$$

$$z + (1/2) = 0 \quad (\text{S-117})$$

$$z = -1/2 \quad (\text{S-118})$$

(c) Compute $h[n]$ for this system. Is this system BIBO-stable?

Solution: We can use the z-transform pair,

$$a^n u[n] \Leftrightarrow \frac{1}{1 - az^{-1}} \quad (\text{S-119})$$

so that,

$$h[n] = (-1/2)^n u[n] \quad (\text{S-120})$$

The system is stable.

(d) Determine $y[n]$ for $x[n] = \delta[n] - 2\delta[n-1] + 4\delta[n-4]$.

Solution: Due to linearity and time invariance,

$$y[n] = h[n] - 2h[n-1] + 4h[n-4] \quad (\text{S-121})$$

$$y[n] = (-1/2)^n u[n] - 2(-1/2)^{(n-1)}u[n-1] + 4(-1/2)^{(n-4)}u[n-4] \quad (\text{S-122})$$

Problem 9:

Given an IIR filter defined by the difference equation:

$$y[n] = \sqrt{2}y[n-1] - y[n-2] + x[n] \tag{9-1}$$

(a) Determine the transfer function $H(z)$ for this system.

Solution:

$$H(z) = \frac{1}{1 - \sqrt{2}z^{-1} + z^{-2}} \tag{S-123}$$

(b) Compute the system poles.

Solution:

$$1 - \sqrt{2}z^{-1} + z^{-2} = 0 \tag{S-124}$$

$$z^2 - \sqrt{2}z + 1 = 0 \tag{S-125}$$

Using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ for } ax^2 + bx + c = 0 \tag{S-126}$$

the system poles are given by,

$$z = \frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{\sqrt{2} \pm \mathbf{j}\sqrt{2}}{2} = e^{\pm \mathbf{j}\pi/4}. \tag{S-127}$$

$$r_1 = e^{\mathbf{j}\pi/4},\ r_2 = e^{-\mathbf{j}\pi/4}. \tag{S-128}$$

(c) Compute $h[n]$ for this system. Is this system BIBO-stable?

Solution: First we compute the partial fraction expansion:

$$\frac{1}{1 - \sqrt{2}z^{-1} + z^{-2}} = \frac{A_1}{1 - r_1 z^{-1}} + \frac{A_2}{1 - r_2 z^{-1}} \tag{S-129}$$

where r_1 and r_2 are given by equation (S-128) above. From the lecture notes,

$$A_1 = H(z)(1 - r_1 z^{-1})\Big|_{z = r_1} = \frac{1}{1 - r_2 z^{-1}}\Bigg|_{z = r_1} = \frac{1}{1 - r_2 r_1^{-1}} \tag{S-130}$$

$$A_2 = H(z)(1 - r_2 z^{-1})\Big|_{z = r_2} = \frac{1}{1 - r_1 z^{-1}}\Bigg|_{z = r_2} = \frac{1}{1 - r_1 r_2^{-1}} \tag{S-131}$$

Plugging (S-128) into (S-130) and (S-131):

$$A_1 = \frac{1}{1 - (e^{-\mathbf{j}\pi/4})(e^{\mathbf{j}\pi/4})^{-1}} = \frac{1}{1 - e^{-\mathbf{j}\pi/2}} = \frac{1}{1 + \mathbf{j}} = \frac{1}{\sqrt{2}e^{\mathbf{j}\pi/4}} = \frac{\sqrt{2}}{2}e^{-\mathbf{j}\pi/4} \tag{S-132}$$

$$A_2 = \frac{1}{1 - (e^{\mathbf{j}\pi/4})(e^{-\mathbf{j}\pi/4})^{-1}} = \frac{1}{1 - e^{\mathbf{j}(\pi/2)}} = \frac{1}{1 - \mathbf{j}} = \frac{1}{\sqrt{2}e^{-\mathbf{j}\pi/4}} = \frac{\sqrt{2}}{2}e^{\mathbf{j}\pi/4} \tag{S-133}$$

Now we can use the z-transform pair,

$$ba^n u[n] \Leftrightarrow \frac{b}{1 - az^{-1}} \tag{S-134}$$

to compute $h[n]$:

$$h[n] = A_1(r_1)^n u[n] + A_2(r_2)^n u[n] \tag{S-135}$$

$$h[n] = \frac{\sqrt{2}}{2} e^{-\mathbf{j}\pi/4}(e^{\mathbf{j}\pi/4})^n u[n] + \frac{\sqrt{2}}{2} e^{\mathbf{j}\pi/4}(e^{-\mathbf{j}\pi/4})^n u[n] \tag{S-136}$$

$$h[n] = \sqrt{2}\left(\frac{(e^{\mathbf{j}\pi/4})^{(n-1)} + (e^{-\mathbf{j}\pi/4})^{(n-1)}}{2}\right) u[n] \tag{S-137}$$

$$h[n] = \sqrt{2}\left(\frac{e^{\mathbf{j}\pi/4(n-1)} + e^{-\mathbf{j}\pi/4(n-1)}}{2}\right) u[n] \tag{S-138}$$

$$h[n] = \sqrt{2}\cos(\pi/4n - \pi/4)u[n]. \tag{S-139}$$

(d) Determine $y[n]$ for $x[n] = \delta[n] - 3\delta[n-1] + 2\delta[n-4]$.

Solution: Due to linearity and time-invariance,

$$y[n] = h[n] - 3h[n-1] + 2h[n-4] \tag{S-140}$$

$$\begin{aligned} y[n] = {} & \sqrt{2}\cos(\pi/4n - \pi/4)u[n] - 3\sqrt{2}\cos(\pi/4(n-1) - \pi/4)u[n-1] + \\ & 2\sqrt{2}\cos(\pi/4(n-4) - \pi/4)u[n-4] \end{aligned} \tag{S-141}$$

$$\begin{aligned} y[n] = {} & \sqrt{2}\cos(\pi/4n - \pi/4)u[n] - 3\sqrt{2}\cos(\pi/4n - \pi/2)u[n-1] + \\ & 2\sqrt{2}\cos(\pi/4 + 3\pi/4)u[n-4] \end{aligned} \tag{S-142}$$

For $n \geq 4$, equation (S-141) simplifies to:

$$y[n] = \sqrt{2}\sin(\pi/4n),\ n \geq 4. \tag{S-143}$$

(S-144)

Problem 10:

Determine the discrete-time signals $x_a[n]$ and $x_b[n]$, respectively, corresponding to the following z-transforms:

$$X_a(z) = \frac{1 - z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} \text{ [part (a)]} \tag{10-1}$$

$$X_b(z) = \frac{1 + z^{-1}}{1 - 0.1z^{-1} - 0.72z^{-2}} \text{ [part (b)]} \tag{10-2}$$

Solution: In the lecture notes, we derived the following transform pair:

$$H(z) = \frac{b_0 + b_1 z^{-1}}{(1 - a_1 z^{-1} - a_2 z^{-2})} \quad \Leftrightarrow \quad h[n] = A_1 r_1^n u[n] + A_2 r_2^n u[n] \tag{10-3}$$

where,

$$r_1 = \frac{a_1 - \sqrt{a_1^2 + 4a_2}}{2},\ r_2 = \frac{a_1 + \sqrt{a_1^2 + 4a_2}}{2},\ A_1 = \frac{b_1 + b_0 r_1}{r_1 - r_2} \text{ and } A_2 = \frac{b_1 + b_0 r_2}{r_2 - r_1}. \tag{S-145}$$

For part (a):

$$b_0 = 1,\ b_1 = -1,\ a_1 = 1/6,\ a_2 = 1/6. \tag{S-146}$$

$$r_1 = \frac{1/6 - \sqrt{(1/6)^2 + 4/6}}{2} = \frac{1/6 - \sqrt{25/36}}{2} = \frac{1/6 - 5/6}{2} = -1/3 \tag{S-147}$$

$$r_2 = \frac{1/6 + \sqrt{(1/6)^2 + 4/6}}{2} = \frac{1/6 + 5/6}{2} = 1/2 \tag{S-148}$$

$$A_1 = \frac{-1 - 1/3}{-1/3 - 1/2} = 8/5 \tag{S-149}$$

$$A_2 = \frac{-1 + 1/2}{1/2 - (-1/3)} = -3/5 \tag{S-150}$$

$$x_a[n] = (8/5)(-1/3)^n u[n] - (3/5)(1/2)^n u[n] \tag{S-151}$$

For part (b):

$$b_0 = 1,\ b_1 = 1,\ a_1 = 0.1,\ a_2 = 0.72. \tag{S-152}$$

$$r_1 = \frac{0.1 - \sqrt{(0.1)^2 + 4(0.72)}}{2} = \frac{0.1 - 1.7}{2} = -0.8 \tag{S-153}$$

$$r_2 = \frac{0.1 + \sqrt{(0.1)^2 + 4(0.72)}}{2} = \frac{0.1 + 1.7}{2} = 0.9 \tag{S-154}$$

$$A_1 = \frac{1 - 0.8}{-0.8 - 0.9} = -2/17 \tag{S-155}$$

$$A_2 = \frac{1 + 0.9}{0.9 - (-0.8)} = 19/17 \tag{S-156}$$

$$x_b[n] = (-2/17)(-0.8)^n u[n] + (19/17)(0.9)^n u[n] \tag{S-157}$$

Problem 11:

An LTI system has the transfer function $H(z)$,

$$H(z) = 1 - 3z^{-2} - 4z^{-4} \quad (11\text{-}1)$$

The input to this system is given by,

$$x[n] = 20 - 20\delta[n] + 20\cos(0.5\pi n + \pi/4), \; -\infty < n < \infty. \quad (11\text{-}2)$$

Determine the output $y[n]$ of the system for all n.

Solution: In the lecture notes, we derived the following transform pair:

$$\delta[n - n_0] \quad \Leftrightarrow \quad z^{-n_0} \quad (\text{S-158})$$

Therefore,

$$h[n] = \delta[n] - 3\delta[n-2] - 4\delta[n-4]. \quad (\text{S-159})$$

This is the impulse response of an FIR filter, whose difference equation is given by,

$$y[n] = x[n] - 3x[n-2] - 4x[n-4]. \quad (\text{S-160})$$

Thus, the output $y[n]$ for the input in equation (11-2) is given by,

$$\begin{aligned} y[n] = {} & 20 - 20\delta[n] + 20\cos(0.5\pi n + \pi/4) - \\ & 3[20 - 20\delta[n-2] + 20\cos(0.5\pi(n-2) + \pi/4)] - \\ & 4[20 - 20\delta[n-4] + 20\cos(0.5\pi(n-4) + \pi/4)] \end{aligned} \quad (\text{S-161})$$

$$\begin{aligned} y[n] = {} & -120 - 20\delta[n] + 60\delta[n-2] + 80\delta[n-4] + 20\cos(0.5\pi n + \pi/4) + \\ & 60\cos(0.5\pi n + \pi/4) - 80\cos(0.5\pi n + \pi/4) \end{aligned} \quad (\text{S-162})$$

$$y[n] = -120 - 20\delta[n] + 60\delta[n-2] + 80\delta[n-4]. \quad (\text{S-163})$$

A more general procedure, that applies to both FIR and IIR filters is to derive the solution for $y[n]$ through the frequency response of the system:

$$H(e^{j\theta}) = H(z)\big|_{z = e^{j\theta}} = 1 - 3e^{-j2\theta} - 4e^{-j4\theta} \quad (11\text{-}3)$$

First, the input $x[n]$ can be broken up into three parts:

$$x[n] = x_1[n] + x_2[n] + x_3[n] \quad (\text{S-164})$$

where,

$$x_1[n] = 20, \; x_2[n] = -20\delta[n] \text{ and } x_3[n] = 20\cos(0.5\pi n + \pi/4). \quad (\text{S-165})$$

We can write the outputs corresponding to $x_1[n]$, $x_2[n]$ and $x_3[n]$:

$$H(e^{j\theta})\big|_{\theta = 0} = 1 - 3 - 4 = -6 \quad (\text{S-166})$$

$$y_1[n] = 20\left|H(e^{j\theta})\right|_{\theta = 0}\cos(0n + \angle H(e^{j\theta})\big|_{\theta = 0}) = 20|-6|\cos(\pi) = -120 \quad (\text{S-167})$$

$$y_2[n] = -20h[n] = -20\delta[n] + 60\delta[n-2] + 80\delta[n-4] \quad (\text{S-168})$$

$$H(e^{j\theta})\big|_{\theta = 0.5\pi} = 1 - 3e^{-j\pi} - 4e^{-j2\pi} = 1 + 3 - 4 = 0 \quad (\text{S-169})$$

$$y_3[n] = 20\left|H(e^{j\theta})\right|_{\theta = 0.5\pi}\cos(0.5\pi n + \angle H(e^{j\theta})\big|_{\theta = 0.5\pi}) = 0 \quad (\text{S-170})$$

Therefore, the total output is given by,

$$y[n] = y_1[n] + y_2[n] + y_3[n] \tag{S-171}$$

$$y[n] = -120 - 20\delta[n] + 60\delta[n-2] + 80\delta[n-4]\,. \tag{S-172}$$

Note that the results in equations (S-163) and (S-172) are equivalent.

Problem 12:

(a) True or False: For the discrete-time system $y[n] = a^n x[n-1]$ the output for an arbitrary input $x[n]$ is given by $y[n] = x[n] * h[n]$, where $h[n]$ is the impulse response of the system.

False: The system is a time-variant system.

(b) True or False: For the discrete-time system $y[n] = y[n-1] + x[n+1]$ the output for an arbitrary input $x[n]$ is given by $y[n] = x[n] * h[n]$, where $h[n]$ is the impulse response of the system.

True. The system is an LTI system.

(c) True or False: For the discrete-time system $y[n] = 2y[n-1] + x[n]$, the frequency response of the system is given by,

$$H(e^{j\theta}) = \frac{1}{1 - 2e^{-j\theta}} \qquad H(z) = \frac{1}{1 - 2z^{-1}} \tag{12-1}$$

False: The pole of $H(z)$ is equal to 2; since the root lies outside the unit circle of the complex plane, the system is unstable. Therefore, equation (12-1) does not represent the frequency response of the system.

(d) True or False: For the discrete-time system $y[n] = (1/2)y[n-1] + x[n]$, the frequency response of the system is given by,

$$H(e^{j\theta}) = \frac{1}{1 - (1/2)e^{-j\theta}} \qquad H(z) = \frac{1}{1 - (1/2)z^{-1}} \tag{12-2}$$

True: The system is stable, so that $H(z)\big|_{z = e^{j\theta}}$ is the frequency response of the system.

(e) True or False: An LTI system with frequency response $H(e^{j\theta}) = u[\theta + 1] - u[\theta - 1]$, $\theta \in [-\pi, \pi]$, is noncausal.

True:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\theta})e^{jn\theta}d\theta = \frac{1}{2\pi}\int_{-1}^{1} e^{jn\theta}d\theta = \frac{e^{jn} - e^{-jn}}{j2n\pi} = \frac{\sin(n)}{n\pi} \tag{S-173}$$

Since $h[n]$ is nonzero for $n < 0$, the system is noncausal.

(f) Give the difference equation for an LTI system with transfer function $H(z)$:

$$H(z) = \frac{1 + z^2 + 3z^4}{z^5} \tag{12-3}$$

$$H(z) = z^{-5} + z^{-3} + 3z^{-1} \quad \Rightarrow \quad h[n] = \delta[n-5] + \delta[n-3] + 3\delta[n-1] \tag{S-174}$$

$$y[n] = x[n-5] + x[n-3] + 3x[n-1] \tag{S-175}$$

(g) Give the difference equation for an LTI system with transfer function $H(z)$:

$$H(z) = \frac{z^2 + 1}{z^2 - z + 1} \tag{12-4}$$

$$H(z) = \frac{1 + z^{-2}}{1 - z^{-1} + z^{-2}} = \frac{Y(z)}{X(z)} \tag{S-176}$$

$$(1 - z^{-1} + z^{-2})Y(z) = (1 + z^{-2})X(z) \tag{S-177}$$

$$Y(z) = z^{-1}Y(z) - z^{-2}Y(z) + X(z) + z^{-2}X(z) \tag{S-178}$$

$$y[n] = y[n-1] - y[n-2] + x[n] + x[n-2] \tag{S-179}$$

Problem 13:

(a) Give the difference equation of an LTI system with frequency response:

$$H(e^{j\theta}) = 1 + \cos(2\theta) + \cos(5\theta) \quad (13\text{-}1)$$

Hint: This problem, when approached correctly, does not involve a lot of computation.

(b) Is your answer in part (a) causal? If not, specify a difference equation for a causal LTI system with the same magnitude frequency response as in equation (13-1).

Solution:

Part (a):

$$H(e^{j\theta}) = 1 + \frac{e^{j2\theta} + e^{-j2\theta}}{2} + \frac{e^{j5\theta} + e^{-j5\theta}}{2} \quad (\text{S-180})$$

$$H(e^{j\theta}) = 1 + \frac{1}{2}e^{j2\theta} + \frac{1}{2}e^{-j2\theta} + \frac{1}{2}e^{j5\theta} + \frac{1}{2}e^{-j5\theta} = \sum_{n=-\infty}^{\infty} h[n]e^{-jn\theta} \quad (\text{S-181})$$

$$h[n] = \delta[n] + \frac{1}{2}\delta[n+2] + \frac{1}{2}\delta[n-2] + \frac{1}{2}\delta[n+5] + \frac{1}{2}\delta[n-5] \quad (\text{S-182})$$

$$h[n] = \frac{1}{2}\delta[n+5] + \frac{1}{2}\delta[n+2] + \delta[n] + \frac{1}{2}\delta[n-2] + \frac{1}{2}\delta[n-5] \quad (\text{S-183})$$

$$y[n] = \frac{1}{2}x[n+5] + \frac{1}{2}x[n+2] + x[n] + \frac{1}{2}x[n-2] + \frac{1}{2}x[n-5] \quad (\text{S-184})$$

Part (b):

Since the system is dependent on future values of the input (i.e. $x[n+5]$, $x[n+2]$), the system is noncausal. We can make the system causal by delaying the system output by five time units:

$$y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-3] + x[n-5] + \frac{1}{2}x[n-7] + \frac{1}{2}x[n-10] \quad (\text{S-185})$$

Systems (S-184) and (S-185) have the same magnitude frequency responses, but different phase frequency responses.

Problem 14:

Consider the following discrete-time system:

$$y[n] = \frac{1}{2}y[n-1] + 2x[n] \tag{14-1}$$

(a) Compute the transfer function $H(z)$ of this system.

(b) Is this system stable?

(c) Give an expression for the output $y[n]$ for the input $x[n] = u[n]$, assuming that $y[n] = 0$, $n < 0$.

(d) Compute the impulse response $h[n]$ of this system.

(e) Compute the frequency response $H(e^{j\theta})$ of this system.

(f) Give an expression for the output $y[n]$ for the following input:

$$x[n] = 10 + 5\delta[n-3] + 2\sin(0.5\pi n + 1),\ -\infty < n < \infty. \tag{14-2}$$

Solution:

Part (a):

$$H(z) = \frac{2}{1-(1/2)z^{-1}} \tag{S-186}$$

Part (b): Since the system pole is equal to $1/2$, which lies inside the unit circle of the complex plane, the system is stable.

Part (c):

$$X(z) = U(z) = \frac{1}{1-z^{-1}} \tag{S-187}$$

$$Y(z) = H(z)X(z) = \frac{2}{(1-(1/2)z^{-1})(1-z^{-1})} \tag{S-188}$$

$$Y(z) = \frac{2}{(1-(1/2)z^{-1})(1-z^{-1})} = \frac{A_1}{(1-(1/2)z^{-1})} + \frac{A_2}{(1-z^{-1})} \tag{S-189}$$

where,

$$A_1 = Y(z)(1-(1/2)z^{-1})\Big|_{z=1/2} = \left.\frac{2}{(1-z^{-1})}\right|_{z=1/2} = \frac{2}{1-2} = -2 \tag{S-190}$$

$$A_2 = Y(z)(1-z^{-1})\Big|_{z=1} = \left.\frac{2}{(1-(1/2)z^{-1})}\right|_{z=1} = \frac{2}{1-(1/2)} = 4 \tag{S-191}$$

Therefore,

$$Y(z) = \frac{-2}{(1-(1/2)z^{-1})} + \frac{4}{(1-z^{-1})} \tag{S-192}$$

$$y[n] = 4u[n] - 2(1/2)^n u[n]. \tag{S-193}$$

Part (d):

$$h[n] = 2(1/2)^n u[n] \tag{S-194}$$

Part (e):

$$H(e^{\mathbf{j}\theta}) = H(z)\big|_{z = e^{j\theta}} = \frac{2}{1 - (1/2)e^{-\mathbf{j}\theta}} \quad \text{(S-195)}$$

Part (f):

Let $x[n] = x_1[n] + x_2[n] + x_3[n]$ where,

$$x_1[n] = 10, \quad \text{(S-196)}$$

$$x_2[n] = 5\delta[n-3], \text{ and,} \quad \text{(S-197)}$$

$$x_3[n] = 2\sin(0.5\pi n + 1). \quad \text{(S-198)}$$

Then $y[n] = y_1[n] + y_2[n] + y_3[n]$ where,

$$y_1[n] = 10\left|H(e^{\mathbf{j}\theta})\right|_{\theta = 0}\cos\left(\angle H(e^{\mathbf{j}\theta})\big|_{\theta = 0}\right), \quad \text{(S-199)}$$

$$y_2[n] = 5h[n-3], \text{ and,} \quad \text{(S-200)}$$

$$y_3[n] = 2\left|H(e^{\mathbf{j}\theta})\right|_{\theta = 0.5\pi}\sin\left(0.5\pi n + 1 + \angle H(e^{\mathbf{j}\theta})\big|_{\theta = 0.5\pi}\right). \quad \text{(S-201)}$$

Note that,

$$H(e^{\mathbf{j}\theta})\big|_{\theta = 0} = \frac{2}{1 - (1/2)} = 4 \text{ so that } \left|H(e^{\mathbf{j}\theta})\right|_{\theta = 0} = 4 \text{ and } \angle H(e^{\mathbf{j}\theta})\big|_{\theta = 0} = 0\text{; and,} \quad \text{(S-202)}$$

$$H(e^{\mathbf{j}\theta})\big|_{\theta = 0.5\pi} = \frac{2}{1 - (1/2)e^{-\mathbf{j}0.5\pi}} = \frac{2}{1 + \mathbf{j}/2} = \frac{4}{2 + \mathbf{j}} \text{ so that,} \quad \text{(S-203)}$$

$$\left|H(e^{\mathbf{j}\theta})\right|_{\theta = 0.5\pi} = 4/\sqrt{5} \text{ and } \angle H(e^{\mathbf{j}\theta})\big|_{\theta = 0.5\pi} = -\operatorname{atan}(1/2) \approx -0.463648. \quad \text{(S-204)}$$

Therefore,

$$y_1[n] = 10(4)\cos(0) = 40, \quad \text{(S-205)}$$

$$y_2[n] = 5h[n-3] = 10(1/2)^{(n-3)}u[n-3], \text{ and,} \quad \text{(S-206)}$$

$$y_3[n] = 2(4/\sqrt{5})\sin(0.5\pi n + 0.536352), \quad \text{(S-207)}$$

and,

$$y[n] = 40 + 10(1/2)^{(n-3)}u[n-3] + 2(4/\sqrt{5})\sin(0.5\pi n + 0.536352). \quad \text{(S-208)}$$

Problem 15:

Consider the LTI system below:

$$y[n] = \frac{1}{2}x[n] - \frac{1}{2}x[n-2]\,. \tag{15-1}$$

(a) Compute the frequency response $H(e^{\mathbf{j}\theta})$ for this system.

(b) Give an analytic expression for $|H(e^{\mathbf{j}\theta})|$. Be sure to simplify your answer as much as possible.

(c) Give an analytic expression for $\angle H(e^{\mathbf{j}\theta})$. Be sure to simplify your answer as much as possible.

(d) Compute the output of this system for the following two inputs:

$$x_1[n] = \cos(\pi n)\,,\ -\infty < n < \infty \tag{15-2}$$

$$x_2[n] = \cos(\pi n/2)\,,\ -\infty < n < \infty\,. \tag{15-3}$$

Solution:

Part (a):

$$H(e^{\mathbf{j}\theta}) = \frac{1}{2} - \frac{1}{2}e^{-\mathbf{j}2\theta} \tag{S-209}$$

Parts (b) and (c):

$$H(e^{\mathbf{j}\theta}) = e^{-\mathbf{j}\theta}\left(\frac{1}{2}e^{\mathbf{j}\theta} - \frac{1}{2}e^{-\mathbf{j}\theta}\right) = \mathbf{j}e^{-\mathbf{j}\theta}\left(\frac{e^{\mathbf{j}\theta} - e^{-\mathbf{j}\theta}}{\mathbf{j}2}\right) = e^{-\mathbf{j}(\theta - \pi/2)}\sin(\theta) \tag{S-210}$$

$$|H(e^{\mathbf{j}\theta})| = |e^{-\mathbf{j}(\theta - \pi/2)}\sin(\theta)| = |\sin(\theta)| \tag{S-211}$$

$$\angle H(e^{\mathbf{j}\theta}) = \begin{cases} -\theta + \pi/2 & \theta \ge 0 \\ -\theta - \pi/2 & \theta < 0 \end{cases} \tag{S-212}$$

Part (d): From above:

$$y_1[n] = |H(e^{\mathbf{j}\theta})|_{\theta = \pi}\cos(\pi n + \angle H(e^{\mathbf{j}\theta})|_{\theta = \pi}) = |\sin(\pi)|\cos(\pi n + \pi/2) = 0 \tag{S-213}$$

$$y_2[n] = |H(e^{\mathbf{j}\theta})|_{\theta = \pi/2}\cos(\pi n + \angle H(e^{\mathbf{j}\theta})|_{\theta = \pi/2}) = |\sin(\pi/2)|\cos(\pi n/2) = \cos(\pi n/2) \tag{S-214}$$

Problem 16:

Assume that you apply the following input $x[n]$ to an LTI system:

$$x[n] = \cos(\pi n/2)u[n] \tag{16-1}$$

and observe the following output $y[n]$:

$$y[n] = 3(2)^{-n}u[n] + 2\cos(\pi n/2)u[n]. \tag{16-2}$$

(a) Using the z-transform pair,

$$a^n u[n] \Leftrightarrow \frac{1}{1 - az^{-1}} \text{ show that } X(z) = \frac{1}{1 + z^{-2}}. \tag{16-3}$$

(b) Compute the transfer function $H(z)$ for this LTI system.

(c) Compute the impulse response $h[n]$ for this LTI system.

(d) Specify the difference equation that describes this LTI system. Is this an FIR or an IIR system?

(e) For the input in equation (16-1), explain why,

$$y[n] = |H(e^{\mathbf{j}\theta})|_{\theta = \pi/2} \cos(\pi n/2 + \angle H(e^{\mathbf{j}\theta})|_{\theta = \pi/2}) \tag{16-4}$$

does not give the output for this LTI system.

(f) Specify $|H(e^{\mathbf{j}\theta})|_{\theta = \pi/2}$ and $\angle H(e^{\mathbf{j}\theta})|_{\theta = \pi/2}$. Hint: This part requires no computation.

Solution:

Part (a):

$$x[n] = (e^{\mathbf{j}\pi n/2} + e^{-\mathbf{j}\pi n/2})u[n] = (e^{\mathbf{j}\pi/2})^n u[n] + (e^{-\mathbf{j}\pi/2})^n u[n] = \mathbf{j}^n u[n] + (-\mathbf{j})^n u[n] \tag{S-215}$$

$$X(z) = \frac{1}{1 - \mathbf{j}z^{-1}} + \frac{1}{1 + \mathbf{j}z^{-1}} = \frac{1}{1 + z^{-2}} \tag{S-216}$$

Part (b):

$$Y(z) = \frac{3}{1 - (1/2)z^{-1}} + \frac{2}{1 + z^{-2}} \tag{S-217}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3(1 + z^{-2})}{1 - (1/2)z^{-1}} + 2 = \frac{3}{1 - (1/2)z^{-1}} + \frac{3z^{-2}}{1 - (1/2)z^{-1}} + 2 \tag{S-218}$$

$$h[n] = 3(1/2)^n u[n] + 3(1/2)^{(n-2)}u[n-2] + 2\delta[n] \tag{S-219}$$

Part (c):

$$h[n] = 3(1/2)^n u[n] + 3(1/2)^{(n-2)}u[n-2] + 2\delta[n] \tag{S-220}$$

Part (d): IIR system. Impulse response $h[n]$ only goes to zero as $n \to \infty$.

Part (e): Note that equation (16-4) is true only for an infinite-length discrete-time sinusoidal input. Since the input in equation (16-1)is zero for $n < 0$, there will be some transient component to the output $y[n]$ (specifically the $3(2)^{-n}u[n]$ part of the response).

Part (f): Note that the nontransient part of the output $y[n]$ is given by $2\cos(\pi n/2)$ for the input $\cos(\pi n/2)$. Therefore,

$$|H(e^{\mathbf{j}\theta})|_{\theta = \pi/2} = 2, \ \angle H(e^{\mathbf{j}\theta})|_{\theta = \pi/2} = 0 \tag{S-221}$$