Fourier Series to Fourier Transform

1. Introduction

In these notes, we continue our discussion of the Fourier series and relate it to the continuous-time Fourier transform through a specific example. We then conclude by looking at the frequency representation (Fourier transform) of several time-limited signals of different duration to observe an important property of the Fourier transform and the frequency spectrum of continuous-time signals.

2. Complex exponential to sinusoidal representation

A. Introduction

Previously, we have computed the Fourier coefficients $X_k$ for some periodic waveforms, and then explicitly derived a real representation (in terms of cosines and sines) for $x(t)$ from the complex exponential form of the Fourier series:

$$ x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kf_0 t} \quad (1) $$

For example, in the lecture #13 notes, we derived the following Fourier coefficients for a triangle wave (symmetric about the vertical axis),

$$ X_k = \begin{cases} \frac{2}{(\pi k)^2} & k = \text{odd} \\ 0 & k = \text{even}, k \neq 0 \\ \frac{1}{2} & k = 0 \end{cases} \quad (2) $$

and converted the complex exponential series,

$$ x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt} \quad (3) $$

to the following sinusoidal representation:

$$ x(t) = 1/2 + \sum_{k=1}^{\infty} \frac{4}{k(\pi k)^2} \cos(2\pi kt), k \in \{1, 3, 5, \ldots\} \quad (4) $$

Here, we will show that once we have computed the Fourier coefficients, we can directly represent real-valued periodic signals $x(t)$ in sinusoidal form using the following relationship:

$$ x(t) = X_0 + 2 \sum_{k=1}^{\infty} |X_k| \cos(2\pi kf_0 t + \angle X_k) \quad (5) $$

B. Proof

To show that equation (5) follows from equation (1), we will use the following property of the Fourier coefficients:

$$ X_{-k} = X_k^{\ast} \quad (6) $$

That is, $X_{-k}$ is the complex conjugate of $X_k$. We begin with equation (1) and use Euler’s equation:

$$ x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kf_0 t} \quad (7) $$
\[ x(t) = \sum_{k = -\infty}^{\infty} X_k \cos(2\pi k f_0 t) + j X_k \sin(2\pi k f_0 t) \]  
(8)

\[ x(t) = X_0 + \sum_{k = 1}^{\infty} X_k \cos(2\pi k f_0 t) + X_{-k} \cos(2\pi(-k)f_0 t) + j(X_k \sin(2\pi k f_0 t) + X_{-k} \sin(2\pi(-k)f_0 t)) \]  
(9)

Now we use equation (6) and the even/odd property of the cosine/sine functions, respectively:

\[ \cos(t) = \cos(-t) \]  
(10)

\[ \sin(t) = -\sin(-t) \]  
(11)

Simplifying equation (9):

\[ x(t) = X_0 + \sum_{k = 1}^{\infty} X_k \cos(2\pi k f_0 t) + X_k^* \cos(2\pi k f_0 t) + j(X_k \sin(2\pi k f_0 t) - X_k^* \sin(2\pi k f_0 t)) \]  
(12)

\[ x(t) = X_0 + \sum_{k = 1}^{\infty} (X_k + X_k^*) \cos(2\pi k f_0 t) + j(X_k - X_k^*) \sin(2\pi k f_0 t) \]  
(13)

Note the following simplifications:

\[ X_k + X_k^* = 2\text{Re}[X_k] \]  
(14)

\[ X_k - X_k^* = 2j\text{Im}[X_k] \]  
(15)

so that equation (13) simplifies to:

\[ x(t) = X_0 + \sum_{k = 1}^{\infty} 2\text{Re}[X_k] \cos(2\pi k f_0 t) - 2\text{Im}[X_k] \sin(2\pi k f_0 t) \]  
(16)

\[ x(t) = X_0 + 2 \sum_{k = 1}^{\infty} \text{Re}[X_k] \cos(2\pi k f_0 t) - \text{Im}[X_k] \sin(2\pi k f_0 t) \]  
(17)

We will now show that for any complex number \( z \), the following relationship is true:

\[ \text{Re}[z] \cos(t) - \text{Im}[z] \sin(t) = |z| \cos(t + \angle z) \]  
(18)

Let \( z = re^{j\theta} \), so that:

\[ \text{Re}[z] = r \cos(\theta) \]  
(19)

\[ \text{Im}[z] = r \sin(\theta) \]  
(20)

\[ |z| = r \]  
(21)

\[ \angle z = \theta \]  
(22)

Plugging (19) through (21) into (18):

\[ r \cos(\theta) \cos(t) - r \sin(\theta) \sin(t) = r \cos(t + \theta) \]  
(23)

\[ \cos(t + \theta) = \cos(\theta) \cos(t) - \sin(\theta) \sin(t) \]  
(24)

Equation (24) is a well known trigonometric identity that we showed to be true (using complex exponentials) in the lecture #12 notes. Having verified equation (18), we can now simplify (17) to the following form:
\[ x(t) = X_0 + 2 \sum_{k=1}^{\infty} |X_k| \cos(2\pi k f_0 t + \angle X_k) \] (25)

C. Examples

In previous notes, we derived the following Fourier coefficients for an odd square wave with period \( T_0 = 1 \) \( (f_0 = 1) \):

\[ X_k = \begin{cases} 
\frac{1/(j\pi k)}{k = \text{odd}} \\
0 & k = \text{even} 
\end{cases} \] (26)

for which we have that:

\[ |X_k| = 1/(\pi k) \rightleftharpoons k \in \{1, 3, 5, \ldots\} \] (27)

\[ \angle X_k = -\pi/2 \rightleftharpoons k \in \{1, 3, 5, \ldots\} \] (28)

Combining (27) and (28) with (25) we get:

\[ x(t) = 2 \sum_{k=\text{odd}} \frac{1}{\pi k} \cos(2\pi k t - \pi/2) \] (29)

\[ x(t) = \sum_{k=\text{odd}} \frac{2}{\pi k} \sin(2\pi k t) \rightleftharpoons k \in \{1, 3, 5, \ldots\}. \] (30)

In the previous notes, we also derived the following Fourier coefficients for an odd sawtooth wave with period \( T_0 = 1 \) \( (f_0 = 1) \):

\[ X_k = \begin{cases} 
\frac{(j(-1)^k)/(2\pi k)}{k \neq 0} \\
0 & k = 0 
\end{cases} \] (31)

for which we have that:

\[ |X_k| = 1/(2\pi k) \rightleftharpoons k > 0 \] (32)

\[ \angle X_k = \begin{cases} 
-\pi/2 & k = \text{odd} \\
\pi/2 & k = \text{even} 
\end{cases} \] (33)

Combining (32) and (33) with (25) we get:

\[ x(t) = 2 \left( \sum_{k=\text{odd}} \frac{1}{(2\pi k)} \cos(2\pi k t - \pi/2) \right) + \sum_{k=\text{even}} \frac{1}{(2\pi k)} \cos(2\pi k t + \pi/2) \] (34)

\[ x(t) = \sum_{k=\text{odd}} \frac{1}{(\pi k)} \sin(2\pi k t) + \sum_{k=\text{even}} \frac{1}{(\pi k)} \sin(2\pi k t) \] (35)

\[ x(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\pi k} \sin(2\pi k t) \rightleftharpoons k \in \{1, 3, 5, \ldots\} \] (36)

Note that in both cases, the results agree with those arrived at explicitly in the lecture #13 notes.
3. Fourier series to Fourier transform

A. Introduction

Here we motivate the continuous Fourier transform as a limiting case of the Fourier series for \( T_0 \to \infty \). We will do this by computing the Fourier series representation of a pulse train waveform \( x(t) \) centered at \( t = 0 \) and varying the period of \( x(t) \). In Figure 1, for example, we plot a pulse train waveform for various periods of increasing width. As \( T_0 \to \infty \), the pulse train waveform approaches a single pulse centered at \( t = 0 \) with width equal to one.

\[
X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)e^{-j2\pi kf_0 t} dt
\]  

(37)

For our problem, the integral in (37) reduces to:

\[
X_k = \frac{1}{T_0} \int_{-1/2}^{1/2} e^{-j2\pi kf_0 t} dt
\]  

(38)

First, we compute \( X_0 \):

\[
X_0 = \frac{1}{T_0} \int_{-1/2}^{1/2} dt = \frac{1}{T_0} \left[ t \right]_{-1/2}^{1/2} = \frac{1}{T_0}
\]  

(39)

Next, we compute \( X_k, k \neq 0 \):
Thus, the Fourier coefficients $X_k$ are given by,

\[
X_k = \frac{\sin(\pi k f_0)}{\pi k} = \frac{\sin(\pi k / T_0)}{\pi k}, \quad k \neq 0, \tag{41}
\]

\[
X_0 = 1 / T_0. \tag{42}
\]

Since $X_k$ is strictly real, we can now plot $X_k$ as a function of frequency $f = kf_0 = k / T_0$ for different values of $T_0$. In Figure 2 below, we plot the Fourier coefficients as a function of frequency for the pulse train waveforms in Figure 1.

![Figure 2](image)

**Figure 2**

C. **Fourier transform**

We now compute the Fourier transform of a rectangular pulse $x_n(t)$ centered at $t = 0$, as plotted in Figure 3 below, which corresponds to a pulse train waveform with $T_0 \to \infty$.

Recall that the Fourier transform of a continuous-time signal $x(t)$ is given by,
For the pulse in Figure 3, this integral is relatively easy to evaluate:

\[ X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \]  

(43)

For the pulse in Figure 3, this integral is relatively easy to evaluate:

\[ X(f) = \int_{-1/2}^{1/2} e^{-j2\pi ft} dt \]  

(44)

\[ X(f) = \left[ \frac{e^{-j2\pi f(1/2)}}{-j2\pi f} - \frac{e^{-j2\pi f(-1/2)}}{-j2\pi f} \right]_{-1/2}^{1/2} \]  

(45)

\[ X(f) = \left( -\frac{1}{j2\pi f} \right) (e^{-j\pi f} - e^{j\pi f}) \]  

(46)

\[ X(f) = \left( \frac{1}{\pi f} \right) \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} = \frac{\sin(\pi f)}{\pi f} \]  

(47)

This function is plotted in Figure 4 below. Note that the frequency spectrum of the single rectangular pulse closely resembles that of the pulse train waveform for large \( T_0 \). In fact, it is possible to compute the Fourier coefficients \( X_k \) of a periodic waveform from the Fourier transform \( X(f) \) of a single period of that waveform, using the following relationship:

\[ X_k = \frac{1}{T_0} X(f)|_{f = k/f_0} \]  

(48)

\[ X(f) \]  

Figure 4
4. An interesting property of the Fourier transform

In this course, we will not delve deeply into the continuous-time Fourier transform and its properties (for much more details on the continuous Fourier transform see, for example, Contemporary Linear Systems: Chapter 5 by R. D. Strum and D. E. Kirk). However, there is one interesting property of the time/frequency representation of signals that is worth exploring further here.

In Figure 5 we plot the Fourier transform $X(f)$ (frequency spectrum) of rectangular pulses $x(t)$ with varying widths. Note that the more spread out the signal is in the time domain, the more compressed it appears in the frequency domain, and vice versa. As a general rule, continuous-time signals that have finite duration in time will have nonzero frequency content throughout the frequency range from positive to negative infinity. Conversely, continuous-time signals that are band-limited in the frequency domain, will have infinite duration throughout time from positive to negative infinity.

![Figure 5](imageURL)

5. Conclusion

The *Mathematica* notebook “ctft.nb” was used to generate the examples in this set of notes. Next time, we will wrap up our discussion of the continuous-time Fourier transform and then transition to our exploration of the discrete-time Fourier transforms.