EEL6667: Homework #1 (Fall 2003)

(6 problems, distributed 9/7 due 9/23)

Starred problems (**) are extra credit.

Instructions:

You may use any mathematical package (e.g. Mathematica, Maple, MathCad, matlab) to help you solve these problems, as long as you turn in a complete printout of your code and runtime output.

Problem 1:

(a) A frame $\{B\}$ is located as follows: initially coincident with a frame $\{A\}$ we rotate $\{B\}$ about $\hat{Z}_B$ by θ degrees and the we rotate the resulting frame about $\hat{X}_B$ by ϕ degrees. Give the rotation matrix which will change the description of vectors from ${}^{B}P$ to ${}^{A}P$. [Craig, Exercise 2.3]

(b) Repeat part (a), except now, let the rotations be about the fixed coordinate axes, $\hat{Z}_A$ and $\hat{X}_A$, respectively.

Problem 2:[Craig, Exercise 2.14]

Develop a general formula to obtain ${}^{A}_{B}T$, where, starting from initial coincidence, $\{B\}$ is rotated by θ about $\hat{K}$, where $\hat{K}$ passes through the point ${}^{A}P$ (not through the origin of $\{A\}$ in general).

Problem 3:[Craig, Exercise 2.15]

$\{A\}$ and $\{B\}$ are frames differing only in orientation. $\{B\}$ is attained as follows: starting coincident with $\{A\}$, $\{B\}$ is rotated by θ radians about unit vector $\hat{K}$. That is,

$$ {}^{A}_{B}R = {}^{A}_{B}R_{\hat{K}}(\theta) \tag{1} $$

Show that

$$ {}^{A}_{B}R = e^{\kappa\theta} \tag{2} $$

where

$$ \kappa = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \tag{3} $$

<u>Hints</u>:

$$ e^{A} = I + \sum_{i=1}^{\infty} (1/i!)A^{i} \tag{4} $$

$$ \frac{de^{\kappa\theta}}{d\theta} = \kappa e^{\kappa\theta} \tag{5} $$

Problem 4:[Craig, Exercise 2.23]

Give an algorithm to construct the definition of a frame ${}^{U}_{A}T$ from three points ${}^{U}P_1$, ${}^{U}P_2$ and ${}^{U}P_3$, where the following is known about these points:

1. ${}^{U}P_1$ is at the origin of $\{A\}$.
2. ${}^{U}P_2$ lies somewhere on the positive $\hat{X}$ axis of $\{A\}$.
3. ${}^{U}P_3$ lies near the positive $\hat{Y}$ axis in the XY plane of $\{A\}$.

Problem 5:

(a) Referring to Figure 1, give the value of ${}^{A}_{B}T$. [Craig, Exercise 2.27]

(b) Referring to Figure 1, give the value of ${}^{A}_{C}T$. [Craig, Exercise 2.28]

(c) Referring to Figure 2, give the value of ${}^{B}_{C}T$. [Craig, Exercise 2.33]

(d) Referring to Figure 2, give the value of ${}^{C}_{A}T$. [Craig, Exercise 2.34]

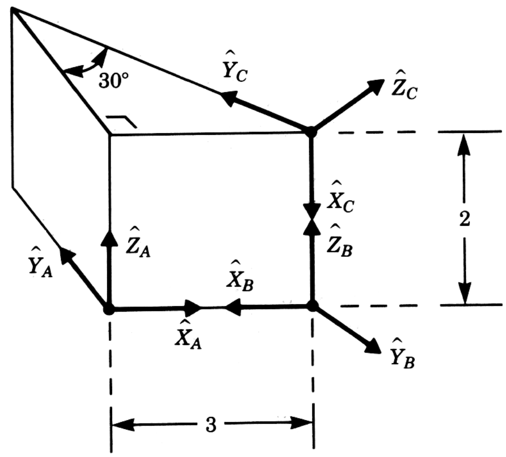

Figure 1

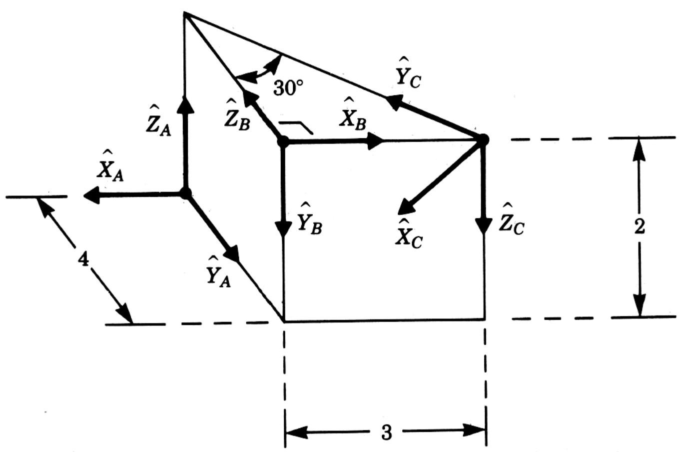

Figure 2

Problem 6:

Given two unit quaternions q and p,

$$q = [s_q, (x_q, y_q, z_q)] \tag{6}$$

$$p = [s_p, (x_p, y_p, z_p)] \tag{7}$$

let us define the following distance metric d:

$$d(q, p) \equiv \min[E(q, p), E(q, -p)] \tag{8}$$

where,

$$E(q, p) \equiv \sqrt{(s_q - s_p)^2 + (x_q - x_p)^2 + (y_q - y_p)^2 + (z_q - z_p)^2} \quad (9)$$

(a) Show that $d(q, p) = d(p, q)$.

(b) Show that $d(q, p) = 0$ if and only if q and p represent the same rotation. Hint: First show that unit quaternions q and $-q$ represent the same rotation.

(c) Let,

$$q = \left[\frac{1}{\sqrt{2}}, \left(0, \frac{1}{\sqrt{2}}, 0\right)\right] \quad (10)$$

$$q' = \frac{1}{4}[\sqrt{6} - 1, (0, \sqrt{6} + 1, \sqrt{2})] \quad (11)$$

$$q'' = \frac{1}{32}[9\sqrt{2} - 4\sqrt{3} - \sqrt{6} - 6, (2\sqrt{3} - 2\sqrt{6} - 4, 11\sqrt{2} + 4\sqrt{3} + 6 - \sqrt{6}, 8\sqrt{6} - 2)] \quad (12)$$

denote three unit quaternions. Compute $d(q, q')$, $d(q', q'')$ and $d(q, q'')$. Do the results obey the triangle inequality?

Note: The triangle inequality is given by,

$$d(a, b) + d(b, c) \geq d(a, c) \quad (13)$$

(d) Assume that q, q' and q'' represent three different rotations Q, Q' and Q''. Derive the equivalent angle-axis $(\theta, \hat{\mathbf{k}})$ representations that describe the following rotations: (1) from Q to Q'; (2) from Q' to Q''; and (3) from Q to Q''. What relationship exists between these results and the computed distances from part (c)?

Note: The rotation between two unit quaternions q and p is given by the quaternion $p^{-1}q$.

(e) Generalize your results from part (d) to show that the distance metric $d(p, q)$ between two quaternions depends only on the angle of rotation θ between the two quaternions, but not the axis of rotation $\hat{\mathbf{k}}$. Give an expression for $d(q, p)$ in terms of θ.

**(f) Use your expression for $d(q, p)$ in terms of θ to show that the triangle inequality in equation (13) holds in general, thus proving that $d(q, p)$ is indeed a *metric*, which must have these properties:

$$d(q, p) = d(p, q) \text{ [part (a)]} \quad (14)$$

$$d(q, p) = 0 \text{ if and only if } q \text{ and } p \text{ represent equivalent rotations [part (b)]} \quad (15)$$

$$d(a, b) + d(b, c) \geq d(a, c) \quad (16)$$