EEL6667: Homework #1 Solutions

Note: "homework1.nb" is a Mathematica notebook that solves many of the problems in this homework.

Problem 1:

(a) See *homework1.nb*.

(b) See *homework1.nb*.

Problem 2:[Craig, Exercise 2.14]

Generalizing example 2.9 in Craig,

$$ {}^{A}_{B}T = {}^{A}_{A'}T\,{}^{A'}_{B'}T\,{}^{B'}_{B}T \tag{1}$$

where frames $\{A'\}$ and $\{B'\}$ are defined on page 53 and illustrated on page 54 of Craig (Figure 2.20). Each individual transform in equation (1) is given by,

$$ {}^{A}_{A'}T = \left[\begin{array}{ccc|c} & I_3 & & {}^{A}P \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{2}$$

$$ {}^{A'}_{B'}T = \left[\begin{array}{ccc|c} & R_{\hat{K}}(\theta) & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{3}$$

$$ {}^{B'}_{B}T = \left[\begin{array}{ccc|c} & I_3 & & -{}^{A}P \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{4}$$

Combining equations (1) through (4):

$$ {}^{A}_{B}T = \left[\begin{array}{ccc|c} & R_{\hat{K}}(\theta) & & [I_3 - R_{\hat{K}}(\theta)]\,{}^{A}P \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{5}$$

Problem 3:[Craig, Exercise 2.15]

We solve this problem by assuming,

$$ {}^{A}_{B}R = e^{\kappa\theta} \tag{6}$$

to be correct and showing that no contradiction results. Now, we differentiate ${}^{A}_{B}R$ two different ways: (1) element by element, and (2) by applying,

$$ \frac{de^{\kappa\theta}}{d\theta} = \kappa e^{\kappa\theta} \tag{7}$$

From homework1.nb, element-wise differentiation yields:

$$ \frac{d({}^{A}_{B}R)}{d\theta} = \begin{bmatrix} (k_x^2-1)\sin(\theta) & k_xk_y\sin(\theta) - k_z\cos(\theta) & k_xk_z\sin(\theta) + k_y\cos(\theta) \\ k_xk_y\sin(\theta) + k_z\cos(\theta) & (k_y^2-1)\sin(\theta) & k_yk_z\sin(\theta) - k_x\cos(\theta) \\ k_xk_z\sin(\theta) - k_y\cos(\theta) & k_yk_z\sin(\theta) + k_x\cos(\theta) & (k_z^2-1)\sin(\theta) \end{bmatrix} \tag{8}$$

Applying equation (7) above, we get that,

$$\frac{d({}^A_BR)}{d\theta} = \kappa\,{}^A_BR \tag{9}$$

which simplifies to,

$$\frac{d({}^A_BR)}{d\theta} = \begin{bmatrix} -(k_y^2+k_z^2)\sin(\theta) & k_xk_y\sin(\theta)-k_z\cos(\theta) & k_xk_z\sin(\theta)+k_y\cos(\theta) \\ k_xk_y\sin(\theta)+k_z\cos(\theta) & -(k_x^2+k_z^2)\sin(\theta) & k_yk_z\sin(\theta)-k_x\cos(\theta) \\ k_xk_z\sin(\theta)-k_y\cos(\theta) & k_yk_z\sin(\theta)+k_x\cos(\theta) & -(k_x^2+k_y^2)\sin(\theta) \end{bmatrix} \tag{10}$$

(See *homework1.nb*.) Note that all the off-diagonal terms in equations (8) and (10) are equivalent, and that the diagonal terms can be made to be equivalent by using the identity,

$$k_x^2+k_y^2+k_z^2 = 1 \tag{11}$$

so that, for example,

$$(k_x^2-1) = -(k_y^2+k_z^2) \tag{12}$$

which shows that the r_{11} element is equivalent in both matrices. The same can be done for r_{22} and r_{33}. (See *homework1.nb*.) Thus, our initial assumption must be correct.

Problem 4:[Craig, Exercise 2.23]

The transformation ${}^U_AT$ is given by,

$$ {}^U_AT = \left[\begin{array}{ccc|c} & {}^U_AR & & {}^UP_{AORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{13}$$

Thus, we need to determine ${}^U_AR$ and ${}^UP_{AORG}$ from the given information. By definition,

$${}^UP_{AORG} = {}^UP_1\,. \tag{14}$$

We also know that,

$${}^U_AR = \begin{bmatrix} {}^U\hat{X}_A & {}^U\hat{Y}_A & {}^U\hat{Z}_A \end{bmatrix} \tag{15}$$

From the problem statement:

$${}^U\hat{X}_A = \frac{({}^UP_2 - {}^UP_1)}{\left\|{}^UP_2 - {}^UP_1\right\|} \tag{16}$$

$${}^U\hat{Z}_A = \frac{{}^U\hat{X}_A \times ({}^UP_3 - {}^UP_1)}{\left\|{}^U\hat{X}_A \times ({}^UP_3 - {}^UP_1)\right\|} \tag{17}$$

$${}^U\hat{Y}_A = {}^U\hat{Z}_A \times {}^U\hat{X}_A \tag{18}$$

Together, equations (13) through (18) completely define ${}^U_AT$.

Problem 5:

(a) From Figure 1,

$$ {}^{A}_{B}R = R_Z(\pi) \tag{19} $$

$$ {}^{A}P_{BORG} = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}^T \tag{20} $$

$$ {}^{A}_{B}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{21} $$

$$ {}^{A}_{B}T = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{22} $$

(See *homework1.nb*.)

(b) From Figure 1,

$$ {}^{B}_{C}R = R_Y(\pi/2)R_X(\pi/2 + \pi/3) \tag{23} $$

$$ {}^{A}_{C}R = {}^{A}_{B}R\,{}^{B}_{C}R \;\; \text{[using } {}^{A}_{B}R \text{ from eq. (19)]} \tag{24} $$

$$ {}^{A}P_{CORG} = \begin{bmatrix} 3 & 0 & 2 \end{bmatrix}^T \tag{25} $$

$$ {}^{A}_{C}T = \left[\begin{array}{ccc|c} & {}^{A}_{C}R & & {}^{A}P_{CORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{26} $$

(See *homework1.nb* for numeric solution.)

(c) Note that Figure 2 (Figure 2.26 in Craig) is inconsistent. The top-surface triangle cannot at the same time have sides of length 3 and 4 respectively, while at the same time have an angle of $30°$. I chose to ignore the indicated length of the longer side, letting it be $3\sqrt{3}$ (instead of 4) to conform with the $30°$ angle specification. Then,

$$ {}^{B}_{C}R = R_Y(\pi)R_X(-\pi/2)R_Z(-\pi/6) \tag{27} $$

$$ {}^{B}P_{CORG} = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}^T \tag{28} $$

$$ {}^{B}_{C}T = \left[\begin{array}{ccc|c} & {}^{B}_{C}R & & {}^{B}P_{CORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{29} $$

(See *homework1.nb* for numeric solution.)

(d) With the same caveat as in part (c) above,

$$ {}^{A}_{B}R = R_Z(\pi)R_X(-\pi/2) \tag{30} $$

$$ {}^{A}_{C}R = {}^{A}_{B}R\,{}^{B}_{C}R \;\; \text{[using } {}^{B}_{C}R \text{ from eq. (27)]} \tag{31} $$

$$ {}^{A}P_{CORG} = \begin{bmatrix} -3 & 3\sqrt{3} & 2 \end{bmatrix}^T \tag{32} $$

$$ {}^{C}_{A}T = \left[\begin{array}{ccc|c} & {}^{A}_{C}R^{T} & & -{}^{A}_{C}R^{T\,A}P_{CORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{33} $$

(See *homework1.nb* for intermediate and numeric solutions.)

Problem 6:

(a) From the definition of the distance metric:

$$ d(q, p) \equiv \min[E(q, p), E(q, -p)] \tag{34} $$

$$ \begin{aligned} d(q, p) &\equiv \min[\sqrt{(s_q - s_p)^2 + (x_q - x_p)^2 + (y_q - y_p)^2 + (z_q - z_p)^2}, \\ &\qquad \sqrt{(s_q + s_p)^2 + (x_q + x_p)^2 + (y_q + y_p)^2 + (z_q + z_p)^2}] \\ &= \min[\sqrt{(s_p - s_q)^2 + (x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2}, \\ &\qquad \sqrt{(s_p + s_q)^2 + (x_p + x_q)^2 + (y_p + y_q)^2 + (z_p + z_q)^2}] \\ &\equiv d(p, q) \end{aligned} \tag{35} $$

(b) For a unit quaternion q,

$$ q = [s, (x, y, z)] \tag{36} $$

the equivalent rotation matrix R is given by,

$$ R = \begin{bmatrix} 1 - 2(y^2 + z^2) & 2(xy - sz) & 2(xz + sy) \\ 2(xy + sz) & 1 - 2(x^2 + z^2) & 2(yz - sx) \\ 2(xz - sy) & 2(yz + sx) & 1 - 2(x^2 + y^2) \end{bmatrix} \tag{37} $$

For a unit quaternion $-q = [-s, (-x, -y, -z)]$, note that you get exactly the same rotation matrix R since the negatives in $-q$ cancel out in every product permutation of $\{-s, -x, -y, -z\}$ that appears in R. Hence, q and $-q$ represent the same rotation.

Now, if $q = p$,

$$ E(q, p) = 0 \tag{38} $$

while if $q = -p$

$$ E(q, -p) = 0\,. \tag{39} $$

Therefore, $d(q, p) = 0$ if $q = p$ or $q = -p$; in other words, if q and p represent the same rotation. From (35) it is self-evident that in all other cases, $d(q, p) \neq 0$.

(c) See *homework1.nb*. The results are:

$$ d(q, q') = \sqrt{2 - \sqrt{3}} \approx 0.51764 \tag{40} $$

$$ d(q', q'') = \sqrt{2 - \sqrt{3}} \approx 0.51764 \tag{41} $$

$$ d(q, q'') = \frac{1}{2}\sqrt{3 + \frac{\sqrt{3}}{2}} \approx 0.98311 \tag{42} $$

It is easy to verify that:

$$ d(q, q') + d(q', q'') \geq d(q, q'') \tag{43} $$

$$d(q', q) + d(q, q'') \geq d(q', q'') \tag{44}$$

$$d(q, q'') + d(q'', q') \geq d(q, q') \tag{45}$$

(d) See *homework1.nb.* Note that,

$$\theta_{qq'} = \theta_{q'q''} = \pi/3 \approx 1.0472 \tag{46}$$

$$\theta_{qq''} \approx 2.0555 \tag{47}$$

where θ_{qp} denotes the angle of rotation from unit quaternion q to p. Observe that the results in (46) and (47) are consistent with the results in (40) through (43). When the angle of rotation between two unit quaternions is equal, so is the distance measure between them. Larger angles of rotation correspond to large distances. The distance measure appears to be independent of $\hat{\mathbf{k}}$.

(e) See *homework1.nb.* Our approach to this problem is as follows. First, we show that the distance metric $d(p, u)$ between an arbitrary unit quaternion,

$$p = [\cos(\theta/2), \sin(\theta/2)(k_x, k_y, k_z)] \tag{48}$$

and the zero-rotation unit quaternion,

$$u = [1, 0, 0, 0] \tag{49}$$

is dependent only on θ (which is the angle of rotation between p and u in this case). So,

$$d(p, u) = \min[\sqrt{(c-1)^2 + (k_x s)^2 + (k_y s)^2 + (k_z s)^2}, \sqrt{(c+1)^2 + (k_x s)^2 + (k_y s)^2 + (k_z s)^2}] \tag{50}$$

where $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$. Since $k_x^2 + k_y^2 + k_z^2 = 1$,

$$d(p, u) = \min[\sqrt{(c-1)^2 + (k_x^2 + k_y^2 + k_z^2)s^2}, \sqrt{(c+1)^2 + (k_x^2 + k_y^2 + k_z^2)s^2}] \tag{51}$$

$$d(p, u) = \min[\sqrt{(c-1)^2 + s^2}, \sqrt{(c+1)^2 + s^2}] \tag{52}$$

$$d(p, u) = \min[\sqrt{2 - 2\cos(\theta/2)}, \sqrt{2 + 2\cos(\theta/2)}] \tag{53}$$

An alternative expression for $d(p, u)$ is given by,

$$d(p, u) = 2 \cdot \min[|\cos(\theta/4)|, |\sin(\theta/4)|] \tag{54}$$

Now, assume that both p and u are multiplied by some arbitrary unit quaternion $q = [s, (x, y, z)]$. Since p and u are rotated by the same rotation q, the relative distance between them should not change, and results(53) and (54) will still hold for $d(qp, q)$. This is verified in *homework1.nb* so that for two arbitrary vectors p and q:

$$d(p, q) = 2 \cdot \min[|\cos(\theta/4)|, |\sin(\theta/4)|] \tag{55}$$

which is plotted in *homework1.nb.*

**(f) Since we know that $d(p, q)$ depends only on θ between p and q, we can rewrite the triangle inequality as:

$$d(\theta_1) + d(\theta_2) \geq d(\theta_1 + \theta_2) \tag{56}$$

Put another way,

$$f(\theta_1, \theta_2) \equiv d(\theta_1) + d(\theta_2) - d(\theta_1 + \theta_2) \geq 0 \ , \forall \theta_1, \theta_2 . \tag{57}$$

where $d(\theta)$ denotes the distance between two unit quaternions separated by an angle θ. In *homework1.nb,* $f(\theta_1, \theta_2)$ is plotted and shown to be greater than zero for $\theta_1, \theta_2 \in [0, 2\pi]$.

(58)