EEL6667: Homework #2 Solutions

Note: "homework2.nb" is an attached Mathematica notebook that solves some of the problems in this homework.

Problem 1:

(a) See Figure 1 for an appropriate assignment of coordinate frames. Note that frame $\{0\}$ is coincident with frame $\{1\}$ for $\theta_1 = 0$. The corresponding DH parameters are (by inspection):

Table 1: DH parameters

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$\pi/2$	L_1	0	θ_2
3	0	L_2	0	θ_3

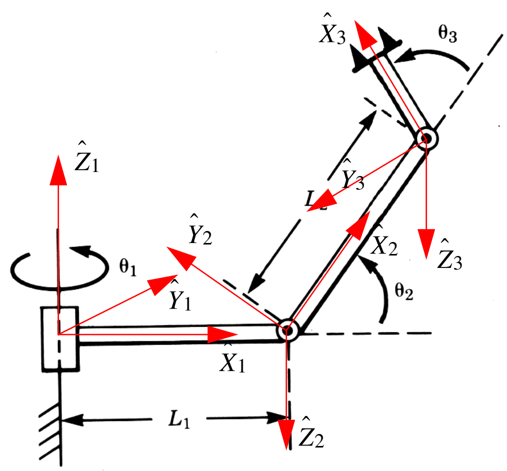

Figure 1

See "homework2.nb," for ${}^0_3T$.

(b) From part (a):

$$ {}^0_3T = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 & c_1(L_1+L_2c_2) \\ s_1c_{23} & -s_1s_{23} & -c_1 & s_1(L_1+L_2c_2) \\ s_{23} & c_{23} & 0 & L_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1) $$

By inspection of equation (1):

$$ \theta_1 = \text{atan}(r_{13}, -r_{23}) \quad (2) $$

$$ \theta_2 = \text{atan}\left(p_z, \pm\sqrt{p_x^2+p_y^2} - L_1\right) \quad \left[\text{Note: } L_2s_2 = p_z,\ L_2c_2 = \pm\sqrt{p_x^2+p_y^2} - L_1\right] \quad (3) $$

$$ \theta_3 = \text{atan}(r_{31}, r_{32}) - \theta_2 \quad (4) $$

Problem 2:

(a) See Figure 2 for an appropriate assignment of coordinate frames. Note that frame $\{0\}$ is coincident with frame $\{1\}$ for $\theta_1 = 0$. The corresponding DH parameters are (by inspection):

Table 2: DH parameters

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$\pi/2$	0	0	θ_2
3	0	L_3	0	θ_3

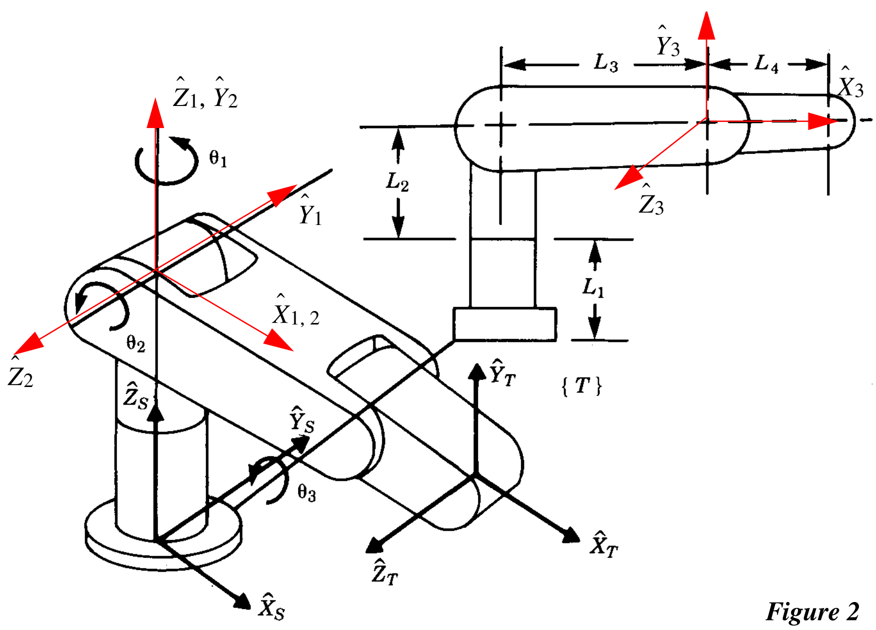

Figure 2

See "homework2.nb," for ${}^0_1T$, ${}^1_2T$ and ${}^2_3T$.

[Note: If you assigned frame $\{0\}$ to be coincident with frame $\{S\}$, then $d_1 = L_1 + L_2$.]

(b) From part (a):

$$ {}^0_3T = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 & L_3c_1c_2 \\ s_1c_{23} & -s_1s_{23} & -c_1 & L_3s_1c_2 \\ s_{23} & c_{23} & 0 & L_3s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5) $$

By inspection of equation (5):

$$ \theta_1 = \operatorname{atan}(r_{13}, -r_{23}) \quad (6) $$

$$ \theta_2 = \operatorname{atan}(p_z, \pm\sqrt{p_x^2 + p_y^2}) \quad \text{[Note: } L_3s_2 = p_z,\ L_3c_2 = \pm\sqrt{p_x^2+p_y^2}\text{]} \quad (7) $$

$$ \theta_3 = \operatorname{atan}(r_{31}, r_{32}) - \theta_2 \quad (8) $$

Problem 3:[Craig, Exercise 3.14]

Consider Figure 3 below.

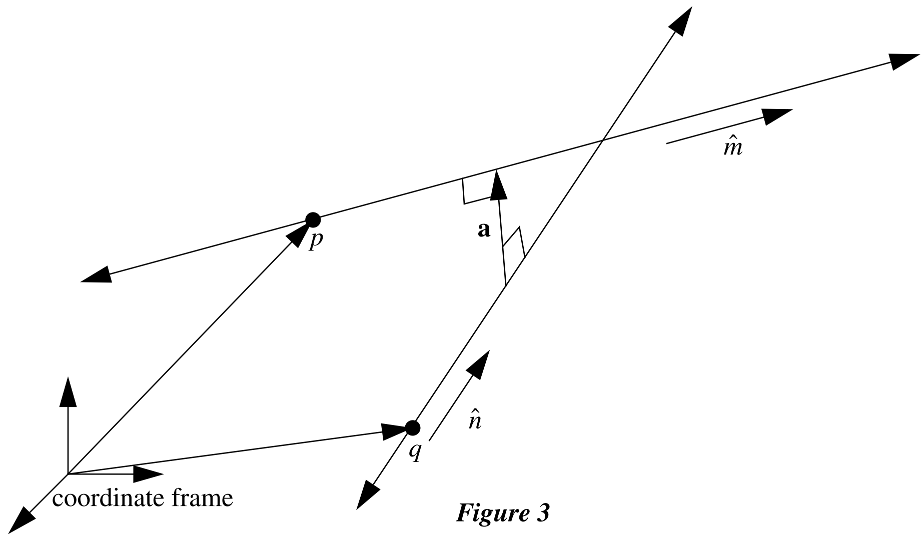

Figure 3

Since $\hat{m}$ and $\hat{n}$ are unit vectors, we know that:

$$\cos(\alpha) = \hat{m} \cdot \hat{n} \tag{9}$$

$$\sin(\alpha) = \|\hat{m} \times \hat{n}\| \tag{10}$$

where α is the angle between $\hat{m}$ and $\hat{n}$. Therefore,

$$\alpha = \mathrm{atan}(\|\hat{m} \times \hat{n}\|, \hat{m} \cdot \hat{n}) . \tag{11}$$

Now, the length a of the vector $\mathbf{a}$ is equal to the length of vector q parallel to the $\mathbf{a}$ vector minus the length of vector p parallel to the $\mathbf{a}$ vector (or the length of vector p parallel to the $\mathbf{a}$ vector minus the length of vector q parallel to the $\mathbf{a}$ vector, whichever turns out to be positive). Since the direction of vector $\mathbf{a}$ is given by $\hat{m} \times \hat{n}$,

$$a = \|q \cdot (\hat{m} \times \hat{n}) - p \cdot (\hat{m} \times \hat{n})\| \tag{12}$$

$$a = \|(q - p) \cdot (\hat{m} \times \hat{n})\| . \tag{13}$$

An equivalent result can be obtained by defining two parameterized lines in space,

$$l_1 = p + \hat{m}s ,\ -\infty < s < \infty \tag{14}$$

$$l_2 = q + \hat{n}t ,\ -\infty < t < \infty \tag{15}$$

so that,

$$a = \min_{s,\,t}\|l_2 - l_1\| \tag{16}$$

The minimization in equation (16) can be accomplished by solving the two equations,

$$\frac{\partial}{\partial s}[(l_2 - l_1) \cdot (l_2 - l_1)] = 0 \tag{17}$$

$$\frac{\partial}{\partial t}[(l_2 - l_1) \cdot (l_2 - l_1)] = 0 \tag{18}$$

for s and t.

Problem 4:[Craig, Exercise 3.16]

See Figure 4 for an appropriate assignment of coordinate frames. Note that frame $\{0\}$ is coincident with frame $\{1\}$ for $\theta_1 = 0$. The corresponding DH parameters are (by inspection):

Table 3: DH parameters

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$\pi/2$	L_1	d_2	0
3	$-\pi/2$	0	0	θ_3

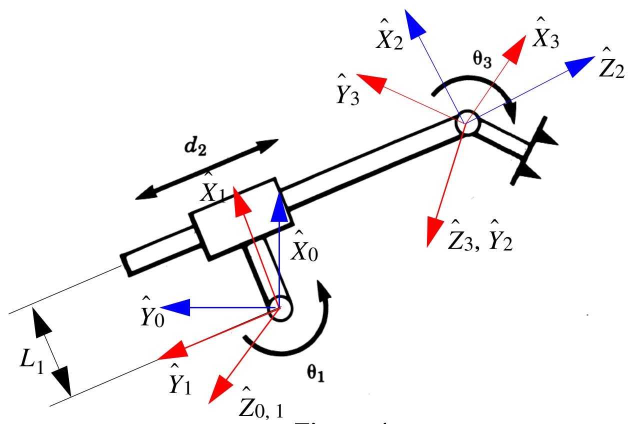

Figure 4

Problem 5:

See "homework2.nb," for a numerically generated plot of the tip workspace ($l_1 = 2$, $l_2 = 1$). Figure 5 below gives a more detailed plot. The different parts of the workspace boundary are numbered and correspond to the following:

$$\text{Number 1: } \theta_2 = 0, \; 0 < \theta_1 < \pi \tag{19}$$

$$\text{Number 2: } \theta_2 = \pi, \; 0 < \theta_1 < \pi \tag{20}$$

$$\text{Number 3: } \theta_1 = \pi, \; 0 < \theta_2 < \pi \tag{21}$$

$$\text{Number 4: } \theta_1 = 0, \; -\pi/2 < \theta_2 < \pi \tag{22}$$

$$\text{Number 5: } \theta_2 = -\pi/2, \; 0 < \theta_1 < x < \pi \text{ for some } x \tag{23}$$

Problem 6:[Craig, Exercise 4.12]

The DH parameters for the non-orthogonal writs mechanism are given below:

Table 4: DH parameters

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1

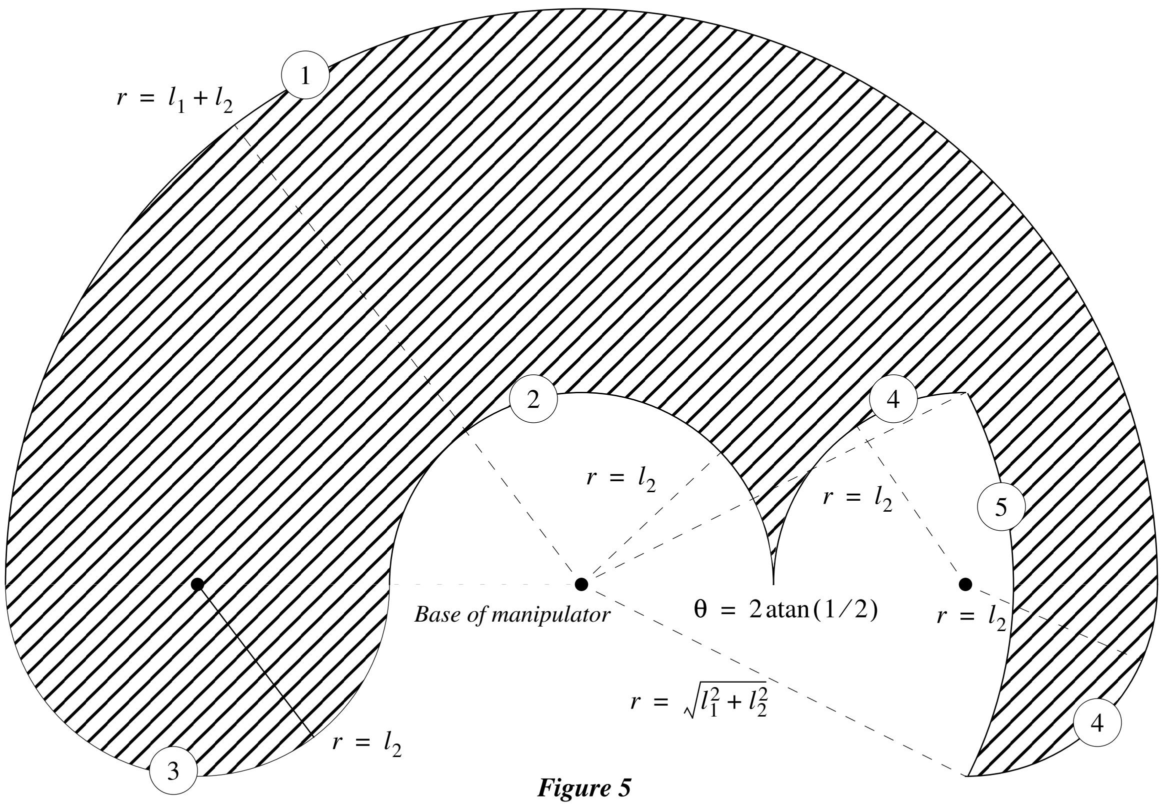

Figure 5

Table 4: DH parameters

i	α_{i-1}	a_{i-1}	d_i	θ_i
2	ϕ	0	0	θ_2
3	$\pi - \phi$	0	0	θ_3

In "homework2.nb," the set of unit vectors $\hat{\mathbf{k}} = \{k_x, k_y, k_z\}$ about which this mechanism can rotate are depicted graphically for $\phi = \pi/2$, $\phi = \pi/4$, and for $\phi \to 0$. From these and other sample values of ϕ, we can generalize to the schematic diagram in Figure 6 below. Figure 6 indicates that no rotations about the vectors $\hat{\mathbf{k}} = \{k_x, k_y, k_z\}$ that lie inside the cones shown with apex angle $\pi - 2\phi$ can be achieved with the non-orthogonal mechanism.

Problem 7:[Craig, Exercise 4.17]

[Note: There is a discrepancy between Problem 7 and [Craig, Exercise 4.17]. In Problem 7, $\alpha_2 = -\pi/4$, while in Exercise 4.17, $\alpha_2 = \pi/4$. Therefore, there are two separate derivations below; the first is for $\alpha_2 = \pi/4$, while the second is for $\alpha_2 = -\pi/4$.]

[Craig, Exercise 4.17] We can use Pieper's solution (Craig, Section 4.6) to solve this problem. Since $a_1 = 0$ (given),

$$r = k_3 \tag{24}$$

where,

$$r \equiv \left\| {}^0P_{4ORG} \right\|^2 = 0^2 + 1^2 + \sqrt{2}^2 = 3 \tag{25}$$

and,

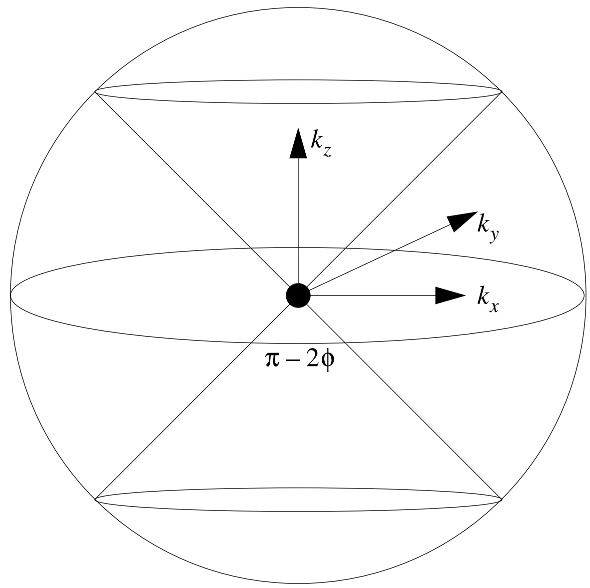

Figure 6

$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 = f_1^2 + f_2^2 + f_3^2 + 2f_3 + 1 \quad [a_1 = 0, d_2 = 1] \tag{26}$$

$$f_1 = a_3 c_3 + d_4 s\alpha_3 s_3 + a_2 = c_3 \quad [a_3 = 1, d_4 = 0, a_2 = 0] \tag{27}$$

$$f_2 = a_3 c\alpha_2 s_3 - d_4 s\alpha_4 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2 = \frac{1}{\sqrt{2}}(s_3 - 1) \quad [a_3 = 1, \alpha_2 = \pi/4, d_4 = 0, d_3 = 1] \tag{28}$$

$$f_3 = a_3 s\alpha_2 s_3 - d_4 s\alpha_4 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2 = \frac{1}{\sqrt{2}}(s_3 + 1) \quad [a_3 = 1, \alpha_2 = \pi/4, d_4 = 0, d_3 = 1]. \tag{29}$$

Combining equations (26) through (29),

$$k_3 = c_3^2 + \frac{1}{2}(s_3 - 1)^2 + \frac{1}{2}(s_3 + 1)^2 + \frac{2}{\sqrt{2}}(s_3 + 1) + 1 \tag{30}$$

$$k_3 = c_3^2 + s_3^2 + 1 + \sqrt{2}(s_3 + 1) + 1 \tag{31}$$

$$k_3 = 1 + 1 + \sqrt{2}s_3 + \sqrt{2} + 1 \tag{32}$$

$$k_3 = 3 + \sqrt{2} + \sqrt{2}s_3 \tag{33}$$

We can now solve for θ_3 by combining equations (24), (25) and (33):

$$3 = 3 + \sqrt{2} + \sqrt{2}s_3 \tag{34}$$

$$s_3 + 1 = 0 \tag{35}$$

$$\sin(\theta_3) = -1 \tag{36}$$

$$\theta_3 = -\pi/2. \tag{37}$$

[Problem 7] We can use Pieper's solution (Craig, Section 4.6) to solve this problem. Since $a_1 = 0$ (given),

$$r = k_3 \tag{38}$$

where,

$$r \equiv \left\| {}^0P_{4ORG} \right\|^2 = 0^2 + 1^2 + \sqrt{2}^2 = 3 \tag{39}$$

and,

$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3 = f_1^2 + f_2^2 + f_3^2 + 2f_3 + 1 \ [a_1 = 0, d_2 = 1] \tag{40}$$

$$f_1 = a_3c_3 + d_4s\alpha_3s_3 + a_2 = c_3 \ [a_3 = 1, d_4 = 0, a_2 = 0] \tag{41}$$

$$f_2 = a_3c\alpha_2s_3 - d_4s\alpha_4c\alpha_2c_3 - d_4s\alpha_2c\alpha_3 - d_3s\alpha_2 = \frac{1}{\sqrt{2}}(s_3 + 1) \ [a_3 = 1, \alpha_2 = -\pi/4, d_4 = 0, d_3 = 1] \tag{42}$$

$$f_3 = a_3s\alpha_2s_3 - d_4s\alpha_4s\alpha_2c_3 + d_4c\alpha_2c\alpha_3 + d_3c\alpha_2 = \frac{1}{\sqrt{2}}(1 - s_3) \ [a_3 = 1, \alpha_2 = -\pi/4, d_4 = 0, d_3 = 1]. \tag{43}$$

Combining equations (40) through (43),

$$k_3 = c_3^2 + \frac{1}{2}(s_3 + 1)^2 + \frac{1}{2}(1 - s_3)^2 + \frac{2}{\sqrt{2}}(1 - s_3) + 1 \tag{44}$$

$$k_3 = c_3^2 + s_3^2 + 1 + \sqrt{2}(1 - s_3) + 1 \tag{45}$$

$$k_3 = 1 + 1 + \sqrt{2} - \sqrt{2}s_3 + 1 \tag{46}$$

$$k_3 = 3 + \sqrt{2} - \sqrt{2}s_3 \tag{47}$$

We can now solve for θ_3 by combining equations (38), (39) and (47):

$$3 = 3 + \sqrt{2} - \sqrt{2}s_3 \tag{48}$$

$$s_3 - 1 = 0 \tag{49}$$

$$\sin(\theta_3) = 1 \tag{50}$$

$$\theta_3 = \pi/2. \tag{51}$$

Problem 8:[Craig, Exercise 4.24]

Given the DH parameters $\{\alpha_{i-1}, a_i, d_i, \theta_i\}$

$$ {}^{i-1}_{\ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_ic\alpha_{i-1} & c\theta_ic\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_is\alpha_{i-1} & c\theta_is\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{52}$$

From equation (52) (by inspection):

$$\theta_i = \operatorname{atan}(-r_{12}, r_{11}), \ \alpha_{i-1} = \operatorname{atan}(-r_{23}, r_{33}), \ a_{i-1} = p_x \text{ and } d_i = \sqrt{p_y^2 + p_z^2}. \tag{53}$$