Introduction to quaternions

Definition: A quaternion q consists of a scalar part s, $s \in \Re$, and a vector part $\mathbf{v} = (x, y, z)$, $\mathbf{v} \in \Re^3$:

$$q \equiv [s, \mathbf{v}]$$

$$q \equiv [s, (x, y, z)]$$

$$q \equiv s + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

where,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\mathbf{ij} = \mathbf{k}$$

$$\mathbf{ji} = -\mathbf{k}$$

Quaternion properties: addition

Definition:

$$q = [s, \mathbf{v}] = [s, (x, y, z)]$$

$$q' = [s', \mathbf{v}'] = [s', (x', y', z')]$$

$$q + q' = [s + s', \mathbf{v} + \mathbf{v}']$$
$$= (s + s') + \mathbf{i}(x + x') + \mathbf{j}(y + y') + \mathbf{k}(z + z')$$

Addition is:

- associative
- commutative

Quaternion properties: multiplication

Definition:

$$qq' = [ss' - \mathbf{v} \cdot \mathbf{v}', \mathbf{v} \times \mathbf{v}' + s\mathbf{v}' + s'\mathbf{v}]$$

Note:

$$\mathbf{a} \cdot \mathbf{b} \equiv a_x b_x + a_y b_y + a_z b_z = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \times \mathbf{b} \equiv (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x) = -\mathbf{b} \times \mathbf{a}$$

Derivation of multiplication

$$qq' \equiv [s, \mathbf{v}][s', \mathbf{v}']$$
$$\equiv (s + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z)(s' + \mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z')$$

$$qq' = ss' + s(\mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z') + \mathbf{i}xs' + \mathbf{i}x(\mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z') +$$
$$(\mathbf{j}ys' + \mathbf{j}y(\mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z') + \mathbf{k}zs' + \mathbf{k}z(\mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z'))$$

$$qq' = ss' + s\mathbf{v}' + s'\mathbf{v} + \mathbf{i}x(\mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z') +$$
$$\mathbf{j}y(\mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z') + \mathbf{k}z(\mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z')$$

$$qq' = ss' + s\mathbf{v}' + s'\mathbf{v} - (xx' + yy' + zz') +$$
$$\mathbf{i}x(\mathbf{j}y' + \mathbf{k}z') + \mathbf{j}y(\mathbf{i}x' + \mathbf{k}z') + \mathbf{k}z(\mathbf{i}x' + \mathbf{j}y')$$

Derivation of multiplication (continued)

$$\begin{aligned} qq' &= ss' + s\mathbf{v}' + s'\mathbf{v} - (xx' + yy' + zz') + \\ &\quad \mathbf{i}x(\mathbf{j}y' + \mathbf{k}z') + \mathbf{j}y(\mathbf{i}x' + \mathbf{k}z') + \mathbf{k}z(\mathbf{i}x' + \mathbf{j}y') \end{aligned}$$

Now:

$$\mathbf{ij} = \mathbf{k}, \; \mathbf{ik} = -\mathbf{j}, \; \mathbf{ji} = -\mathbf{k}, \; \mathbf{jk} = \mathbf{i}, \; \mathbf{ki} = \mathbf{j}, \; \mathbf{kj} = -\mathbf{i}$$

So:

$$\begin{aligned} qq' &= ss' + s\mathbf{v}' + s'\mathbf{v} - (xx' + yy' + zz') + \\ &\quad xy'\mathbf{k} - xz'\mathbf{j} - yx'\mathbf{k} + yz'\mathbf{i} + zx'\mathbf{j} - zy'\mathbf{i} \end{aligned}$$

Derivation of multiplication (continued)

$$\begin{aligned} qq' &= ss' + s\mathbf{v}' + s'\mathbf{v} - (xx' + yy' + zz') + \\ &\quad xy'\mathbf{k} - xz'\mathbf{j} - yx'\mathbf{k} + yz'\mathbf{i} + zx'\mathbf{j} - zy'\mathbf{i} \end{aligned}$$

$$\begin{aligned} qq' &= ss' + s\mathbf{v}' + s'\mathbf{v} - (xx' + yy' + zz') + \\ &\quad (yz' - zy')\mathbf{i} + (zx' - xz')\mathbf{j} + (xy' - yx')\mathbf{k} \end{aligned}$$

$$\begin{aligned} qq' &= ss' + s\mathbf{v}' + s'\mathbf{v} - \mathbf{v} \cdot \mathbf{v}' + \mathbf{v} \times \mathbf{v}' \\ &= [ss' - \mathbf{v} \cdot \mathbf{v}', \mathbf{v} \times \mathbf{v}' + s\mathbf{v}' + s'\mathbf{v}] \end{aligned}$$

Quaternion properties

Conjugate:

$$\|q\| = \sqrt{qq^*}$$

Note:

$$\begin{aligned} qq^* &= [s^2 - (\mathbf{v} \cdot (-\mathbf{v})), \mathbf{v} \times (-\mathbf{v}) + s\mathbf{v} + s(-\mathbf{v})] \\ &= [s^2 + \mathbf{v} \cdot \mathbf{v}, \mathbf{0}] \end{aligned}$$

So:

$$\|q\| = \sqrt{s^2 + x^2 + y^2 + z^2}$$

Unit quaternion: $\|q\| = 1$

Quaternion properties

Inverse:

$$q^{-1} = q^* / \|q\|^2$$

Note:

$$qq^{-1} = q^{-1}q = qq^* / \|q\|^2 = \|q\|^2 / \|q\|^2 = [1, \mathbf{0}]$$

Quaternion rotations

Any unit quaternion can be expressed as:

$$q = [\cos(\theta/2), \sin(\theta/2)\hat{\mathbf{k}}]$$

where,

$$\hat{\mathbf{k}} = (k_x, k_y, k_z).$$

Such a unit quaternion represents a rotation of θ about the vector $\hat{\mathbf{k}}$.

Quaternion rotation

Let,

$$\mathbf{p} = (p_x, p_y, p_z)$$

denote the Cartesian coordinates of a point in 3-space.

Rotation of $\mathbf{p}$ by θ about the vector $\hat{\mathbf{k}}$ to $\mathbf{p}'$:

$$p = [0, \mathbf{p}]$$

$$p' = [0, \mathbf{p}']$$

$$p' = qpq^*$$

Quaternion rotation: angle-axis equivalence

$$p' = qpq^*$$

Remember:

$$qq' = [ss' - \mathbf{v} \cdot \mathbf{v}', \mathbf{v} \times \mathbf{v}' + s\mathbf{v}' + s'\mathbf{v}]$$

So:

$$p' = [s, \mathbf{v}][0, \mathbf{p}][s, -\mathbf{v}]$$

$$p' = [-\mathbf{v} \cdot \mathbf{p}, \mathbf{v} \times \mathbf{p} + s\mathbf{p}][s, -\mathbf{v}]$$

$$p' = \begin{matrix} [-s(\mathbf{v} \cdot \mathbf{p}) + (\mathbf{v} \times \mathbf{p}) \cdot \mathbf{v} + s(\mathbf{p} \cdot \mathbf{v}), \\ -(\mathbf{v} \times \mathbf{p}) \times \mathbf{v} - s(\mathbf{p} \times \mathbf{v}) + s(\mathbf{v} \times \mathbf{p}) + s^2\mathbf{p} + \mathbf{v}(\mathbf{v} \cdot \mathbf{p})] \end{matrix}$$

Quaternion rotation: angle-axis equivalence

$$p' = \begin{matrix} [-s(\mathbf{v} \cdot \mathbf{p}) + (\mathbf{v} \times \mathbf{p}) \cdot \mathbf{v} + s(\mathbf{p} \cdot \mathbf{v}), \\ -(\mathbf{v} \times \mathbf{p}) \times \mathbf{v} - s(\mathbf{p} \times \mathbf{v}) + s(\mathbf{v} \times \mathbf{p}) + s^2\mathbf{p} + \mathbf{v}(\mathbf{v} \cdot \mathbf{p})] \end{matrix}$$

Scalar part:

$$-s(\mathbf{v} \cdot \mathbf{p}) + (\mathbf{v} \times \mathbf{p}) \cdot \mathbf{v} + s(\mathbf{p} \cdot \mathbf{v}) = 0$$

(Why? *Mathematica*)

Vector part I:

$$-s(\mathbf{p} \times \mathbf{v}) + s(\mathbf{v} \times \mathbf{p}) = 2s(\mathbf{v} \times \mathbf{p})$$

So:

$$p' = [0, -(\mathbf{v} \times \mathbf{p}) \times \mathbf{v} + 2s(\mathbf{v} \times \mathbf{p}) + s^2\mathbf{p} + \mathbf{v}(\mathbf{v} \cdot \mathbf{p})]$$

Quaternion rotation: angle-axis equivalence

$$p' = [0, -(\mathbf{v} \times \mathbf{p}) \times \mathbf{v} + 2s(\mathbf{v} \times \mathbf{p}) + s^2\mathbf{p} + \mathbf{v}(\mathbf{v} \cdot \mathbf{p})]$$

Vector part II:

$$(\mathbf{v} \times \mathbf{p}) \times \mathbf{v} = (\mathbf{v} \cdot \mathbf{v})\mathbf{p} - (\mathbf{v} \cdot \mathbf{p})\mathbf{v}$$ (Why? *Mathematica*)

So:

$$p' = [0, (s^2 - \mathbf{v} \cdot \mathbf{v})\mathbf{p} + 2s(\mathbf{v} \times \mathbf{p}) + 2(\mathbf{v} \cdot \mathbf{p})\mathbf{v}]$$

$$\mathbf{p}' = (s^2 - \mathbf{v} \cdot \mathbf{v})\mathbf{p} + 2s(\mathbf{v} \times \mathbf{p}) + 2(\mathbf{v} \cdot \mathbf{p})\mathbf{v}$$

Quaternion rotation: angle-axis equivalence

$$\mathbf{p}' = (s^2 - \mathbf{v} \cdot \mathbf{v})\mathbf{p} + 2s(\mathbf{v} \times \mathbf{p}) + 2(\mathbf{v} \cdot \mathbf{p})\mathbf{v}$$

Remember unit quaternion:

$$q = [\cos(\theta/2), \sin(\theta/2)\hat{\mathbf{k}}]$$

So:

$$\mathbf{p}' = [\cos^2(\theta/2) - \sin^2(\theta/2)]\mathbf{p} + 2\cos(\theta/2)\sin(\theta/2)\hat{\mathbf{k}} \times \mathbf{p} + 2\sin^2(\theta/2)(\hat{\mathbf{k}} \cdot \mathbf{p})\hat{\mathbf{k}}$$

Quaternion rotation: angle-axis equivalence

$$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$$

$$\sin\theta = 2\cos(\theta/2)\sin(\theta/2)$$

$$(1 - \cos\theta) = (1 - \cos^2(\theta/2) + \sin^2(\theta/2)) = 2\sin^2(\theta/2)$$

So:

$$\mathbf{p}' = [\cos^2(\theta/2) - \sin^2(\theta/2)]\mathbf{p} + 2\cos(\theta/2)\sin(\theta/2)\hat{\mathbf{k}} \times \mathbf{p} + 2\sin^2(\theta/2)(\hat{\mathbf{k}} \cdot \mathbf{p})\hat{\mathbf{k}}$$

$$\mathbf{p}' = (\cos\theta)\mathbf{p} + (\sin\theta)\hat{\mathbf{k}} \times \mathbf{p} + (1 - \cos\theta)(\hat{\mathbf{k}} \cdot \mathbf{p})\hat{\mathbf{k}}$$

Quaternion rotation: angle-axis equivalence

$$\mathbf{p}' = (\cos\theta)\mathbf{p} + (\sin\theta)\hat{\mathbf{k}} \times \mathbf{p} + (1 - \cos\theta)(\hat{\mathbf{k}} \cdot \mathbf{p})\hat{\mathbf{k}}$$

$$\mathbf{p}' = (c\theta)\mathbf{p} + (s\theta)\hat{\mathbf{k}} \times \mathbf{p} + (v\theta)(\hat{\mathbf{k}} \cdot \mathbf{p})\hat{\mathbf{k}}$$

where,

$$c\theta = \cos\theta$$

$$s\theta = \sin\theta$$

$$v\theta = 1 - \cos\theta$$

Equivalent to $\mathbf{p}' = R_{\hat{\mathbf{k}}}(\theta)\mathbf{p}$?

(Best answered by *Mathematica*.)

Quaternion rotation: angle-axis equivalence

Answer from *Mathematica*:

$$\mathbf{p}' = R_{\hat{\mathbf{k}}}(\theta)\mathbf{p}$$

$$R_{\hat{\mathbf{k}}}(\theta) = \begin{bmatrix} (k_x^2 v\theta + c\theta) & (k_x k_y v\theta - k_z s\theta) & (k_x k_z v\theta + k_y s\theta) \\ (k_x k_y v\theta + k_z s\theta) & (k_y^2 v\theta + c\theta) & (k_y k_z v\theta - k_x s\theta) \\ (k_x k_z v\theta - k_y s\theta) & (k_y k_z v\theta + k_x s\theta) & (k_y^2 v\theta + c\theta) \end{bmatrix}$$

Conclusion: same as derived earlier!

Quaternion rotation: summary

Rotation through quaternions:

$$p' = qpq^*$$

Rodrigues formula:

$$\mathbf{p}' = (c\theta)\mathbf{p} + (s\theta)\hat{\mathbf{k}} \times \mathbf{p} + (v\theta)(\hat{\mathbf{k}} \cdot \mathbf{p})\hat{\mathbf{k}}$$

Composite quaternion rotations

Assume two rotations: q_o followed by q_1 :

$$p' = q_0 p q_0^* \quad \text{(first rotation)}$$

$$p'' = q_1 p' q_1^* \quad \text{(second rotation)}$$

$$p'' = q_1 (q_0 p q_0^*) q_1^*$$

$$p'' = (q_1 q_0) p (q_0^* q_1^*)$$

$$p'' = (q_1 q_0) p (q_0^* q_1^*)$$

$$p'' = (q_1 q_0) p (q_1 q_0)^*$$

Conjugate multiplication

$$qq' = [ss' - \mathbf{v} \cdot \mathbf{v}', \mathbf{v} \times \mathbf{v}' + s\mathbf{v}' + s'\mathbf{v}] \quad \text{(multiplication)}$$

$$\begin{aligned} (q_0^* q_1^*) &= [s_0, -\mathbf{v}_0][s_1, -\mathbf{v}_1] \\ &= [s_0 s_1 - \mathbf{v}_0 \cdot \mathbf{v}_1, \mathbf{v}_0 \times \mathbf{v}_1 - s_0 \mathbf{v}_0 - s_1 \mathbf{v}_1] \end{aligned}$$

$$\begin{aligned} (q_1 q_0)^* &= ([s_1, \mathbf{v}_1][s_0, \mathbf{v}_0])^* \\ &= [s_1 s_0 - \mathbf{v}_1 \cdot \mathbf{v}_0, \mathbf{v}_1 \times \mathbf{v}_0 + s_0 \mathbf{v}_1 + s_1 \mathbf{v}_0]^* \\ &= [s_1 s_0 - \mathbf{v}_1 \cdot \mathbf{v}_0, -(\mathbf{v}_1 \times \mathbf{v}_0 + s_0 \mathbf{v}_1 + s_1 \mathbf{v}_0)] \\ &= [s_0 s_1 - \mathbf{v}_0 \cdot \mathbf{v}_1, \mathbf{v}_0 \times \mathbf{v}_1 - s_0 \mathbf{v}_0 - s_1 \mathbf{v}_1] \end{aligned}$$

$$(q_0^* q_1^*) = (q_1 q_0)^*$$

Quaternion to angle-axis representation

Unit quaternion:

$$q = [s, \mathbf{v}]$$

$$q = [\cos(\theta/2), \sin(\theta/2)\hat{\mathbf{k}}]$$

So:

$$\theta = 2\,\mathrm{atan}(\|\mathbf{v}\|, s)$$

$$\hat{\mathbf{k}} = \mathbf{v}/\|\mathbf{v}\|$$

Quaternion to rotation matrix

Given:

$$q = [s, (x, y, z)]$$

what is

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \;?$$

Quaternion to rotation matrix

$$\mathbf{p}' = (s^2 - \mathbf{v}\cdot\mathbf{v})\mathbf{p} + 2s(\mathbf{v}\times\mathbf{p}) + 2(\mathbf{v}\cdot\mathbf{p})\mathbf{v}$$

Now:

$$(s^2 - \mathbf{v}\cdot\mathbf{v})\mathbf{p} = (s^2 - x^2 - y^2 - z^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{p}$$

$$2s(\mathbf{v}\times\mathbf{p}) = 2s\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}\mathbf{p} \qquad 2(\mathbf{v}\cdot\mathbf{p})\mathbf{v} = 2\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}\mathbf{p}$$

(...see *Mathematica*)

Quaternion to rotation matrix (cont.)

$$\mathbf{p}' = \left((s^2 - x^2 - y^2 - z^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2s\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} + 2\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} \right)\mathbf{p}$$

Quaternion to rotation matrix (cont.)

$$\mathbf{p}' = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-sz) & 2(xz+sy) \\ 2(xy+sz) & 1-2(x^2+z^2) & 2(yz-sx) \\ 2(xz-sy) & 2(yz+sx) & 1-2(x^2+y^2) \end{bmatrix} \mathbf{p}$$

Therefore, a unit quaternion $q = [s, (x, y, z)]$ corresponds to:

$$R = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-sz) & 2(xz+sy) \\ 2(xy+sz) & 1-2(x^2+z^2) & 2(yz-sx) \\ 2(xz-sy) & 2(yz+sx) & 1-2(x^2+y^2) \end{bmatrix}$$

Rotation matrix to quaternion

Given:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

what is

$$q = [s, (x, y, z)] \text{ ?}$$

Rotation matrix to quaternion

$$R = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-sz) & 2(xz+sy) \\ 2(xy+sz) & 1-2(x^2+z^2) & 2(yz-sx) \\ 2(xz-sy) & 2(yz+sx) & 1-2(x^2+y^2) \end{bmatrix}$$

or

$$R = \begin{bmatrix} s^2+x^2-y^2-z^2 & 2(xy-sz) & 2(xz+sy) \\ 2(xy+sz) & s^2-x^2+y^2-z^2 & 2(yz-sx) \\ 2(xz-sy) & 2(yz+sx) & s^2-x^2-y^2+z^2 \end{bmatrix}$$

Rotation matrix to quaternion

$$R = \begin{bmatrix} s^2+x^2-y^2-z^2 & 2(xy-sz) & 2(xz+sy) \\ 2(xy+sz) & s^2-x^2+y^2-z^2 & 2(yz-sx) \\ 2(xz-sy) & 2(yz+sx) & s^2-x^2-y^2+z^2 \end{bmatrix}$$

So:

$$r_{11} + r_{22} + r_{33} = 3s^2 - x^2 - y^2 - z^2$$

$$r_{11} + r_{22} + r_{33} + 1 = 3s^2 - x^2 - y^2 - z^2 + (s^2 + x^2 + y^2 + z^2)$$

$$r_{11} + r_{22} + r_{33} + 1 = 4s^2$$

Rotation matrix to quaternion

$$r_{11} + r_{22} + r_{33} + 1 = 4s^2$$

$$s^2 = (1/4)(1 + r_{11} + r_{22} + r_{33})$$

Similarly:

$$x^2 = (1/4)(1 + r_{11} - r_{22} - r_{33})$$

$$y^2 = (1/4)(1 - r_{11} + r_{22} - r_{33})$$

$$z^2 = (1/4)(1 - r_{11} - r_{22} + r_{33})$$

Rotation matrix to quaternion

$$R = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-sz) & 2(xz+sy) \\ 2(xy+sz) & 1-2(x^2+z^2) & 2(yz-sx) \\ 2(xz-sy) & 2(yz+sx) & 1-2(x^2+y^2) \end{bmatrix}$$

$$r_{32} - r_{23} = 2(yz+sx) - 2(yz-sx)$$
$$= 4sx$$

$$sx = (1/4)(r_{32} - r_{23}) \quad sy = (1/4)(r_{13} - r_{31})$$

$$sz = (1/4)(r_{21} - r_{12}) \quad xy = (1/4)(r_{12} + r_{21})$$

$$xz = (1/4)(r_{13} + r_{31}) \quad yz = (1/4)(r_{23} + r_{32})$$

Rotation matrix to quaternion

$$s^2 = (1/4)(1 + r_{11} + r_{22} + r_{33})$$

$$x^2 = (1/4)(1 + r_{11} - r_{22} - r_{33})$$

$$y^2 = (1/4)(1 - r_{11} + r_{22} - r_{33})$$

$$z^2 = (1/4)(1 - r_{11} - r_{22} + r_{33})$$

$$sx = (1/4)(r_{32} - r_{23}) \quad sy = (1/4)(r_{13} - r_{31})$$

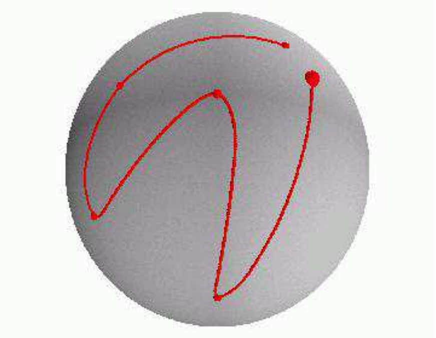
$$sz = (1/4)(r_{21} - r_{12}) \quad xy = (1/4)(r_{12} + r_{21})$$

$$xz = (1/4)(r_{13} + r_{31}) \quad yz = (1/4)(r_{23} + r_{32})$$

Now what?

Quaternion advantages

1. Do not suffer from singularities, as 3-angle conventions do.
2. Most compact way of representing rotations without redundancy or singularity.



Quaternion advantages (cont.)

3. Distance metric between rotations:

$$d(q, p) \equiv \min[E(q, p), E(q, -p)]$$

$$E(q, p) \equiv \sqrt{(s_q - s_p)^2 + (x_q - x_p)^2 + (y_q - y_p)^2 + (z_q - z_p)^2}$$

4. Finite-precision re-normalization of compound rotations.

5. For simulation purposes, easy to generate a uniformly random distribution of rotations in quaternion space.

6. Computation of shortest-path, smooth, continuous-velocity trajectories between two rotations (represented as unit quaternions).

Trajectory generation between rotations

Linear interpolation (Lerp):

$$\text{Lerp}(q_0, q_1, h) = \frac{f(q_0, q_1, h)}{\|f(q_0, q_1, h)\|}, \quad h \in [0, 1],$$

where,

$$f(q_0, q_1, h) = q_0(1 - h) + q_1 h .$$

Trajectory generation between rotations

Spherical Linear interpolation (Slerp):

- Constant velocity
- Shortest path

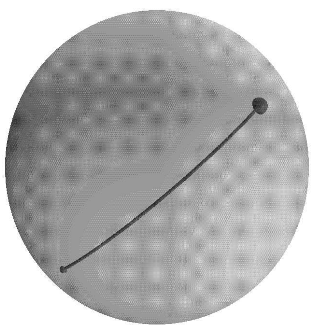
$$\text{Slerp}(q_0, q_1, h) = (q_1 q_0^{-1})^h q_0, \quad h \in [0, 1]$$

or equivalently as,

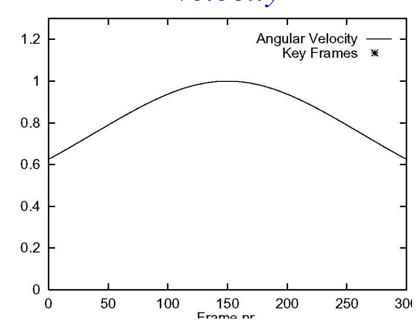
$$\text{Slerp}(q_0, q_1, h) = (q_1 q_0^*)^h q_0, \quad h \in [0, 1] \quad \text{(why?)}$$

So what's the difference?

Linear interpolation (Lerp)

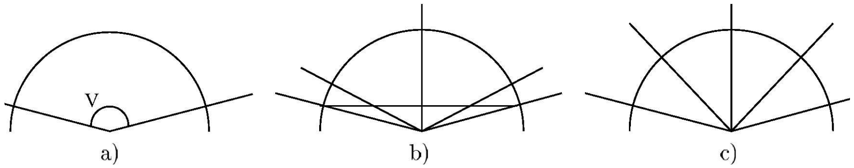


velocity

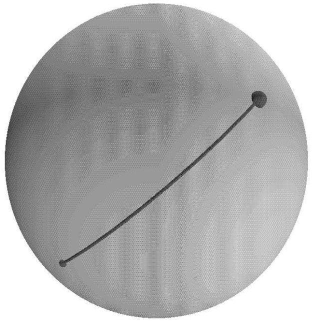


- Nonconstant velocity...

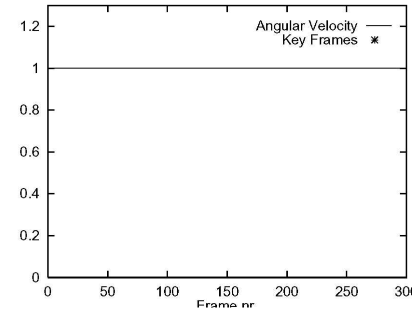
Why nonconstant velocity?



Spherical linear interpolation (Slerp)



velocity



- Constant velocity...