Iterative General Dynamic Model for Serial-Link Manipulators

1. Introduction

In this set of notes, we are going to develop a method for computing a general dynamic model for serial-link manipulators. This dynamic model will relate the set of torques or forces τ required at the joints of the manipulator (torques for revolute joints, forces for prismatic joints) to achieve a particular set of joint positions, velocities and accelerations $(\Theta, \dot{\Theta}, \ddot{\Theta})$:

$$\tau = h(\Theta, \dot{\Theta}, \ddot{\Theta}) \tag{1}$$

Note that in equation (1), τ, Θ, $\dot{\Theta}$ and $\ddot{\Theta}$ are all $n \times 1$ vectors, where n is the number of joints in the manipulator, and h is some nonlinear function. In order to derive such a dynamic model, we will do the following:

1. Relate linear and angular accelerations of coordinate frames with respect to one another.
2. Generalize the basic laws of motions to three dimensions and apply those laws of motion to the serial configuration of manipulators.
3. Extend the concept of moments of inertia to three dimensions (inertia tensor). *[This part of the discussion will not be part of this set of notes, but will be handled elsewhere.]*

2. Accelerations between coordinate frames

A. Basic definitions

The linear acceleration ${}^B\dot{V}_Q$ of a point Q with respect to some coordinate frame $\{B\}$ is defined as the time derivative of the linear velocity ${}^BV_Q$:

$${}^B\dot{V}_Q = \frac{d}{dt}({}^BV_Q) = \lim_{\Delta t \to 0}\left(\frac{{}^BV_Q(t+\Delta t) - {}^BV_Q(t)}{\Delta t}\right) \tag{2}$$

Note that this definitions is similar to that of linear velocity itself (from Chapter 5):

$${}^BV_Q = \frac{d}{dt}({}^BQ) = \lim_{\Delta t \to 0}\left(\frac{{}^BQ(t+\Delta t) - {}^BQ(t)}{\Delta t}\right) \tag{3}$$

Similarly, the angular acceleration ${}^A\dot{\Omega}_B$ of coordinate frame $\{A\}$ with respect to coordinate frame $\{B\}$ is defined as the time derivative of the angular acceleration ${}^A\Omega_B$:

$${}^A\dot{\Omega}_B = \frac{d}{dt}({}^A\Omega_B) = \lim_{\Delta t \to 0}\left(\frac{{}^A\Omega_B(t+\Delta t) - {}^A\Omega_B(t)}{\Delta t}\right) \tag{4}$$

B. Notation

We define the following short-hand notation for the linear and angular accelerations of some coordinate frame $\{A\}$ with respect to a *fixed* universal reference frame $\{U\}$:

$\dot{v}_A = {}^U\dot{V}_{AORG}$ (similar to $v_A = {}^UV_{AORG}$ from Chapter 5) (5)

$\dot{\omega}_A = {}^U\dot{\Omega}_A$ (similar to $\omega_A = {}^U\Omega_A$ from Chapter 5) (6)

C. Linear accelerations between coordinate frames

In Chapter 5, we derived the following important equation for two coordinate frames $\{A\}$ and $\{B\}$ with *coincident* origins (i.e. no translation between $\{A\}$ and $\{B\}$):

${}^AV_Q = {}^A_BR({}^BV_Q) + {}^A\Omega_B \times [{}^A_BR({}^BQ)]$ ($\{A\}$ and $\{B\}$ origins *coincident*). (7)

Let us rewrite equation (7) to establish an important formula for the time derivative of a rotation matrix times a vector:

$$ {}^{A}V_{Q} = {}^{A}_{B}R({}^{B}V_{Q}) + {}^{A}\Omega_{B} \times [{}^{A}_{B}R({}^{B}Q)] \tag{8}$$

$$ \frac{d}{dt}[{}^{A}_{B}R({}^{B}Q)] = {}^{A}_{B}R({}^{B}\dot{Q}) + {}^{A}\Omega_{B} \times [{}^{A}_{B}R({}^{B}Q)] \tag{9}$$

$$ \frac{d}{dt}[{}^{A}_{B}R({}^{B}Q)] = {}^{A}_{B}R({}^{B}\dot{Q}) + {}^{A}\Omega_{B} \times [{}^{A}_{B}R({}^{B}Q)] \tag{10}$$

Note that equation (10) gives us a general formula for the time derivative of RQ where R is some rotation matrix and Q is some vector.

Now, let us differentiate equation (8) with respect to time:

$$ {}^{A}V_{Q} = {}^{A}_{B}R({}^{B}V_{Q}) + {}^{A}\Omega_{B} \times [{}^{A}_{B}R({}^{B}Q)] \tag{11}$$

$$ {}^{A}\dot{V}_{Q} = \frac{d}{dt}[{}^{A}_{B}R({}^{B}V_{Q})] + {}^{A}\dot{\Omega}_{B} \times [{}^{A}_{B}R({}^{B}Q)] + {}^{A}\Omega_{B} \times \frac{d}{dt}[{}^{A}_{B}R({}^{B}Q)] \tag{12}$$

In equation (12) we made use of the following identity for any 3-space vectors Q and P:

$$ \frac{d}{dt}(Q \times P) = \frac{dQ}{dt} \times P + Q \times \frac{dP}{dt} = \dot{Q} \times P + Q \times \dot{P}. \tag{13}$$

Let us now substitute equation (10) into equation (12):

$$ {}^{A}\dot{V}_{Q} = \frac{d}{dt}[{}^{A}_{B}R({}^{B}V_{Q})] + {}^{A}\dot{\Omega}_{B} \times [{}^{A}_{B}R({}^{B}Q)] + {}^{A}\Omega_{B} \times \frac{d}{dt}[{}^{A}_{B}R({}^{B}Q)] \tag{14}$$

$$ \begin{aligned} {}^{A}\dot{V}_{Q} = {} & \left\{ {}^{A}_{B}R({}^{B}\dot{V}_{Q}) + {}^{A}\Omega_{B} \times [{}^{A}_{B}R({}^{B}V_{Q})] \right\} + {}^{A}\dot{\Omega}_{B} \times [{}^{A}_{B}R({}^{B}Q)] + \\ & {}^{A}\Omega_{B} \times \left\{ {}^{A}_{B}R({}^{B}\dot{Q}) + {}^{A}\Omega_{B} \times [{}^{A}_{B}R({}^{B}Q)] \right\} \end{aligned} \tag{15}$$

In equation (15), let ${}^{B}\dot{Q} = {}^{B}V_{Q}$ so that:

$$ \begin{aligned} {}^{A}\dot{V}_{Q} = {} & \left\{ {}^{A}_{B}R({}^{B}\dot{V}_{Q}) + {}^{A}\Omega_{B} \times [{}^{A}_{B}R({}^{B}V_{Q})] \right\} + {}^{A}\dot{\Omega}_{B} \times [{}^{A}_{B}R({}^{B}Q)] + \\ & {}^{A}\Omega_{B} \times \left\{ {}^{A}_{B}R({}^{B}V_{Q}) + {}^{A}\Omega_{B} \times [{}^{A}_{B}R({}^{B}Q)] \right\} \end{aligned} \tag{16}$$

Note that we can combine two cross-product terms in equation (16),

$$ \begin{aligned} {}^{A}\dot{V}_{Q} = {} & \left\{ {}^{A}_{B}R({}^{B}\dot{V}_{Q}) + {}^{A}\Omega_{B} \times [{}^{A}_{B}R({}^{B}V_{Q})] \right\} + {}^{A}\dot{\Omega}_{B} \times [{}^{A}_{B}R({}^{B}Q)] + \\ & {}^{A}\Omega_{B} \times \left\{ {}^{A}_{B}R({}^{B}V_{Q}) + {}^{A}\Omega_{B} \times [{}^{A}_{B}R({}^{B}Q)] \right\} \end{aligned} \tag{17}$$

$$ {}^{A}\dot{V}_{Q} = {}^{A}_{B}R({}^{B}\dot{V}_{Q}) + 2\,{}^{A}\Omega_{B} \times [{}^{A}_{B}R({}^{B}V_{Q})] + {}^{A}\dot{\Omega}_{B} \times [{}^{A}_{B}R({}^{B}Q)] + {}^{A}\Omega_{B} \times [{}^{A}\Omega_{B} \times {}^{A}_{B}R({}^{B}Q)] \tag{18}$$

$$ {}^A\dot{V}_Q = {}^A_BR({}^B\dot{V}_Q) + 2\,{}^A\Omega_B \times [{}^A_BR({}^BV_Q)] + {}^A\dot{\Omega}_B \times [{}^A_BR({}^BQ)] + {}^A\Omega_B \times [{}^A\Omega_B \times {}^A_BR({}^BQ)] \tag{19} $$

Equation (19) gives the angular acceleration ${}^A\dot{V}_Q$ for a vector Q defined with respect to coordinate frame $\{B\}$, when the origins of coordinate frames $\{A\}$ and $\{B\}$ are coincident with one another. If the origin of coordinate frame $\{B\}$ is accelerating with respect to $\{A\}$ equation (19) is easily modified to include that additional linear acceleration:

$$ {}^A\dot{V}_Q = {}^A\dot{V}_{BORG} + {}^A_BR({}^B\dot{V}_Q) + 2\,{}^A\Omega_B \times [{}^A_BR({}^BV_Q)] + {}^A\dot{\Omega}_B \times [{}^A_BR({}^BQ)] + {}^A\Omega_B \times [{}^A\Omega_B \times {}^A_BR({}^BQ)] \tag{20} $$

$$ {}^A\dot{V}_Q = {}^A\dot{V}_{BORG} + {}^A_BR({}^B\dot{V}_Q) + 2\,{}^A\Omega_B \times [{}^A_BR({}^BV_Q)] + {}^A\dot{\Omega}_B \times [{}^A_BR({}^BQ)] + {}^A\Omega_B \times [{}^A\Omega_B \times {}^A_BR({}^BQ)] \tag{21} $$

Equation (21) allows for the possibility that vector ${}^BQ$ is moving with respect to coordinate frame $\{B\}$. Let us now consider a more restrictive case — namely, that vector ${}^BQ$ is fixed with respect to coordinate frame $\{B\}$. In other words, we will assume that there is movement only between coordinate frames $\{A\}$ and $\{B\}$ (and not within coordinate frame $\{B\}$), so that:

$$ {}^BV_Q = 0 \tag{22} $$

$$ {}^B\dot{V}_Q = 0 \tag{23} $$

This assumption simplifies equation (21) substantially:

$$ {}^A\dot{V}_Q = {}^A\dot{V}_{BORG} + {}^A_BR({}^B\dot{V}_Q) + 2\,{}^A\Omega_B \times [{}^A_BR({}^BV_Q)] + {}^A\dot{\Omega}_B \times [{}^A_BR({}^BQ)] + {}^A\Omega_B \times [{}^A\Omega_B \times {}^A_BR({}^BQ)] \tag{24} $$

$$ {}^A\dot{V}_Q = {}^A\dot{V}_{BORG} + {}^A\dot{\Omega}_B \times [{}^A_BR({}^BQ)] + {}^A\Omega_B \times [{}^A\Omega_B \times {}^A_BR({}^BQ)],\quad {}^BV_Q = {}^B\dot{V}_Q = 0 \tag{25} $$

D. Angular accelerations between coordinate frames

Now, let us consider angular accelerations between different coordinate frames. Let us begin with the relationship of angular velocities between three coordinate frames $\{A\}$, $\{B\}$ and $\{C\}$:

$$ {}^A\Omega_C = {}^A\Omega_B + {}^A_BR({}^B\Omega_C) \tag{26} $$

Differentiating (26) and again substituting equation (10), we get:

$$ {}^A\dot{\Omega}_C = {}^A\dot{\Omega}_B + \frac{d}{dt}[{}^A_BR({}^B\Omega_C)] \tag{27} $$

$$ {}^A\dot{\Omega}_C = {}^A\dot{\Omega}_B + {}^A_BR({}^B\dot{\Omega}_C) + {}^A\Omega_B \times [{}^A_BR({}^B\Omega_C)] \tag{28} $$

$$ {}^A\dot{\Omega}_C = {}^A\dot{\Omega}_B + {}^A_BR({}^B\dot{\Omega}_C) + {}^A\Omega_B \times [{}^A_BR({}^B\Omega_C)] \tag{29} $$

E. Summary of results

Below, we summarize our results on linear and angular accelerations:

$$ {}^A\dot{V}_Q = {}^A\dot{V}_{BORG} + {}^A_BR({}^B\dot{V}_Q) + 2\,{}^A\Omega_B \times [{}^A_BR({}^BV_Q)] + {}^A\dot{\Omega}_B \times [{}^A_BR({}^BQ)] + {}^A\Omega_B \times [{}^A\Omega_B \times {}^A_BR({}^BQ)] \tag{30} $$

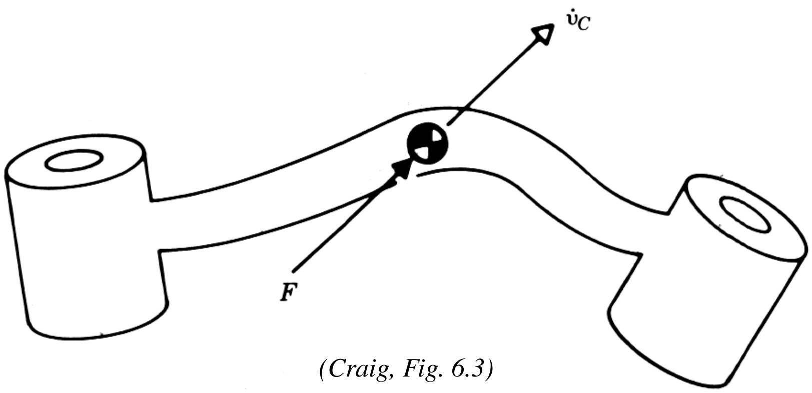
$$ {}^A\dot{V}_Q = {}^A\dot{V}_{BORG} + {}^A\dot{\Omega}_B \times [{}^A_BR({}^BQ)] + {}^A\Omega_B \times [{}^A\Omega_B \times {}^A_BR({}^BQ)],\quad {}^BV_Q = {}^B\dot{V}_Q = 0 \tag{31} $$

$$ {}^A\dot{\Omega}_C = {}^A\dot{\Omega}_B + {}^A_BR({}^B\dot{\Omega}_C) + {}^A\Omega_B \times [{}^A_BR({}^B\Omega_C)] \tag{32} $$

3. Basic equations of motion

A. Newton's law

Consider Figure 1 below, which depicts a rigid body, whose center of mass is accelerating with acceleration $\dot{v}_C$ under a net force F acting on the body.



(Craig, Fig. 6.3)

Figure 1: Rigid body, whose center of mass is accelerating under the action of a net force *F*.

Newton's second law of motion relates F and $\dot{v}_C$:

$$F = m\dot{v}_C \tag{33}$$

where,

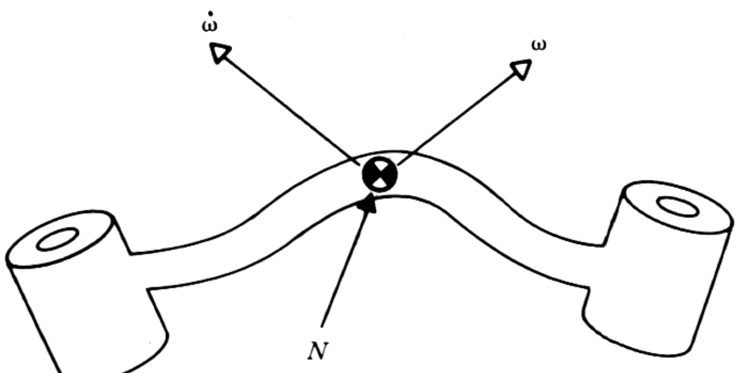
F = net force acting on the body, (34)

m = mass of the body, and, (35)

$\dot{v}_C$ = acceleration of the center of mass of the body. (36)

B. Euler's law

Consider Figure 2 below, which depicts a rigid body, which is rotating with angular velocity ω and angular acceleration $\dot{\omega}$ under a net moment N acting on the body.



(Craig, Fig. 6.4)

Figure 2: Rigid body, which is rotating under the action of a net moment *N*.

Euler's law of rotational motion for rigid bodies relates N and $(\omega, \dot{\omega})$:

$$N = {}^{C}I\dot{\omega} + \omega \times {}^{C}I\omega \tag{37}$$

where,

N = net moment acting on the body, (38)

${}^{C}I = 3 \times 3$ inertia tensor, written with respect to coordinate frame $\{C\}$ (at center of mass), (39)

ω = angular velocity of the body, and, (40)

$\dot{\omega}$ = angular acceleration of the body. (41)

C. Dynamic modeling

Given the results of Section 2 on accelerations, and the basic laws of motion in the previous two sub-sections, we will now proceed as follows in deriving the dynamic model for a serial-link manipulator:

$$\tau = h(\Theta, \dot{\Theta}, \ddot{\Theta}) \tag{42}$$

We will assume that the joint positions Θ, joint velocities $\dot{\Theta}$ and joint accelerations $\ddot{\Theta}$ are known, so that we can compute the corresponding joint torques/forces τ required to achieve the known joint trajectory. We then break down the development of the dynamic model in equation (42) into three main tasks:

1. Outward propagation of velocities and acceleration from the base coordinate frame $\{0\}$ to the end-effector coordinate frame $\{N\}$.
2. Newton's and Euler's equations of motion from the base coordinate frame $\{0\}$ to the end-effector coordinate frame $\{N\}$.
3. Inward propagation of force balance and moment balance equations from coordinate frame $\{N\}$ to coordinate frame $\{1\}$.

4. Propagation of velocities and accelerations

A. Angular velocities and accelerations

In Chapter 5, we developed the following relationship for angular velocities between consecutive links:

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \quad (\textit{velocity}\text{ propagation}) \tag{43}$$

Now, we want to develop an analogous relationship for *angular accelerations*:

$${}^{i+1}\dot{\omega}_{i+1} = g({}^{i}\omega_i, {}^{i}\dot{\omega}_i) \tag{44}$$

where $g(\ )$ represents some functional mapping. To do this, we will apply the general relationship that we developed in Section 2 for the propagation of angular accelerations:

$${}^{A}\Omega_C = {}^{A}\Omega_B + {}^{A}_{B}R({}^{B}\Omega_C) \tag{45}$$

$${}^{A}\dot{\Omega}_C = {}^{A}\dot{\Omega}_B + {}^{A}_{B}R({}^{B}\dot{\Omega}_C) + {}^{A}\Omega_B \times [{}^{A}_{B}R({}^{B}\Omega_C)]. \tag{46}$$

First, we will get equation (43) into the same form as equation (45). Recall that ${}^{i+1}\omega_{i+1}$ and ${}^{i}\omega_i$ are shorthand notation, and can be written less compactly as,

$${}^{i}\omega_i = {}^{i}_{U}R({}^{U}\Omega_i) \text{ and } {}^{i+1}\omega_{i+1} = {}^{i+1}_{U}R({}^{U}\Omega_{i+1}). \tag{47}$$

Also note that,

$$\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}_{i}R({}^{i}\Omega_{i+1})\,. \tag{48}$$

Equation (48) requires some explanation. The right-hand side of (48) denotes the angular velocity of coordinate frame $\{i+1\}$ with respect to $\{i\}$ expressed in the $\{i+1\}$ coordinate frame. Now, think about the left-hand side of (48). The angular velocity between frames $\{i\}$ and $\{i+1\}$ is given exactly by the joint rate $\dot{\theta}_{i+1}$ oriented along the $\hat{Z}_{i+1}$ axis, and that the left-hand side of (48) is expressed in terms of the $\{i+1\}$ coordinate frame. Substituting (47) and (48) into equation (43),

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \tag{49}$$

$${}^{i+1}_{U}R({}^{U}\Omega_{i+1}) = {}^{i+1}_{i}R\,{}^{i}_{U}R({}^{U}\Omega_i) + {}^{i+1}_{i}R({}^{i}\Omega_{i+1}) \tag{50}$$

$${}^{i+1}_{U}R({}^{U}\Omega_{i+1}) = {}^{i+1}_{i}R\,{}^{i}_{U}R({}^{U}\Omega_i) + {}^{i+1}_{i}R({}^{i}\Omega_{i+1}) \tag{51}$$

Now, let $i = B$ and $i+1 = C$ so that equation (51) becomes:

$${}^{C}_{U}R({}^{U}\Omega_C) = {}^{C}_{B}R\,{}^{B}_{U}R({}^{U}\Omega_B) + {}^{C}_{B}R({}^{B}\Omega_C) \tag{52}$$

$${}^{C}_{U}R({}^{U}\Omega_C) = {}^{C}_{U}R({}^{U}\Omega_B) + {}^{C}_{B}R({}^{B}\Omega_C) \tag{53}$$

Multiplying equation (53) by ${}^{U}_{C}R$ and letting $U = A$,

$${}^{U}_{C}R\,{}^{C}_{U}R({}^{U}\Omega_C) = {}^{U}_{C}R\,{}^{C}_{U}R({}^{U}\Omega_B) + {}^{U}_{C}R\,{}^{C}_{B}R({}^{B}\Omega_C) \tag{54}$$

$${}^{U}\Omega_C = {}^{U}\Omega_B + {}^{U}_{B}R({}^{B}\Omega_C) \tag{55}$$

$${}^{A}\Omega_C = {}^{A}\Omega_B + {}^{A}_{B}R({}^{B}\Omega_C) \tag{56}$$

Note that we have now transformed equation (43) into (45), which means that equation (46) can be used to propagate angular accelerations from one link to the next for serial-link manipulators.

In summary,

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \Leftrightarrow {}^{A}\Omega_C = {}^{A}\Omega_B + {}^{A}_{B}R({}^{B}\Omega_C) \tag{57}$$

with substitutions:

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{U}R({}^{U}\Omega_{i+1}) \tag{58}$$

$${}^{i}\omega_i = {}^{i}_{U}R({}^{U}\Omega_i) \tag{59}$$

$$\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}_{i}R({}^{i}\Omega_{i+1})\,, \tag{60}$$

$$i = B\,,\ i+1 = C \text{ and } U = A\,. \tag{61}$$

From equation (46), angular accelerations are propagated by,

$${}^{A}\dot{\Omega}_C = {}^{A}\dot{\Omega}_B + {}^{A}_{B}R({}^{B}\dot{\Omega}_C) + {}^{A}\Omega_B \times [{}^{A}_{B}R({}^{B}\Omega_C)] \tag{62}$$

Let us now transform this relationship into link notation by reverse substitution. First let $B = i$, $C = i+1$ and $A = U$, so that:

$${}^{U}\dot{\Omega}_{i+1} = {}^{U}\dot{\Omega}_i + {}^{U}_{i}R({}^{i}\dot{\Omega}_{i+1}) + {}^{U}\Omega_i \times [{}^{U}_{i}R({}^{i}\Omega_{i+1})]\,. \tag{63}$$

Multiply equation (63) by ${}^{i+1}_{U}R$:

$$ {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_{i+1} = {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_{i} + {}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}\dot{\Omega}_{i+1}) + {}^{i+1}_{U}R\left\{{}^{U}\Omega_{i} \times [{}^{U}_{i}R({}^{i}\Omega_{i+1})]\right\} \tag{64}$$

In order to further develop equation (64), we will need the following identity: for any 3-space vectors Q, P and rotation matrix R,

$$ R(Q \times P) = (RQ) \times (RP) . \tag{65}$$

Thus, we can modify the last term of (64) using (65):

$$ {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_{i+1} = {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_{i} + {}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}\dot{\Omega}_{i+1}) + {}^{i+1}_{U}R\left\{{}^{U}\Omega_{i} \times [{}^{U}_{i}R({}^{i}\Omega_{i+1})]\right\} \tag{66}$$

$$ {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_{i+1} = {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_{i} + {}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}\dot{\Omega}_{i+1}) + [{}^{i+1}_{U}R\,{}^{U}\Omega_{i}] \times [{}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}\Omega_{i+1})] \tag{67}$$

Using,

$$ {}^{i+1}_{U}R\,{}^{U}_{i}R = {}^{i+1}_{i}R \text{ and } {}^{i+1}_{U}R = {}^{i+1}_{i}R\,{}^{i}_{U}R , \tag{68}$$

equation (67) further simplifies to:

$$ {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_{i+1} = {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_{i} + {}^{i+1}_{i}R\,{}^{i}\dot{\Omega}_{i+1} + [{}^{i+1}_{U}R\,{}^{U}\Omega_{i}] \times ({}^{i+1}_{i}R\,{}^{i}\Omega_{i+1}) \tag{69}$$

$$ {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_{i+1} = {}^{i+1}_{i}R({}^{i}_{U}R\,{}^{U}\dot{\Omega}_{i}) + {}^{i+1}_{i}R\,{}^{i}\dot{\Omega}_{i+1} + [{}^{i+1}_{i}R({}^{i}_{U}R\,{}^{U}\Omega_{i})] \times ({}^{i+1}_{i}R\,{}^{i}\Omega_{i+1}) \tag{70}$$

Finally, we make the following substitutions into equation (70):

$$ {}^{i}_{U}R\,{}^{U}\Omega_{i} = {}^{i}\omega_{i} \text{ (by definition)} \tag{71}$$

$$ {}^{i}_{U}R\,{}^{U}\dot{\Omega}_{i} = {}^{i}\dot{\omega}_{i} \text{ and } {}^{i+1}_{U}R({}^{U}\dot{\Omega}_{i+1}) = {}^{i+1}\dot{\omega}_{i+1} \text{ (by definition)} \tag{72}$$

$$ {}^{i+1}_{i}R\,{}^{i}\Omega_{i+1} = \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \text{ [previously explained in discussion following (48)]} \tag{73}$$

$$ {}^{i+1}_{i}R\,{}^{i}\dot{\Omega}_{i+1} = \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \text{ [analogous to (73) above]} \tag{74}$$

These substitutions result in:

$$ {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_{i+1} = {}^{i+1}_{i}R({}^{i}_{U}R\,{}^{U}\dot{\Omega}_{i}) + {}^{i+1}_{i}R\,{}^{i}\dot{\Omega}_{i+1} + [{}^{i+1}_{i}R({}^{i}_{U}R\,{}^{U}\Omega_{i})] \times ({}^{i+1}_{i}R\,{}^{i}\Omega_{i+1}) \tag{75}$$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_{i} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + ({}^{i+1}_{i}R\,{}^{i}\omega_{i}) \times (\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}) \tag{76}$$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_{i} + ({}^{i+1}_{i}R\,{}^{i}\omega_{i}) \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \tag{77}$$

B. Angular velocities and accelerations: summary

Summarizing the results developed in the previous section, angular velocities and accelerations are propagated from link i to link $i+1$ using the following two equations (for revolute joints):

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \text{ [from (43)]} \tag{78}$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + ({}^{i+1}_{i}R\,{}^{i}\omega_i) \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \text{ [from (77)]} \quad (79)$$

Note that equations (78) and (79) allow us to compute the angular velocity $^{i+1}\omega_{i+1}$ of link frame $\{i+1\}$ in terms of the angular velocity $^{i}\omega_i$ of link frame $\{i\}$ and $\dot{\theta}_{i+1}$; and the angular acceleration $^{i+1}\dot{\omega}_{i+1}$ of link frame $\{i+1\}$ in terms of the angular acceleration $^{i}\dot{\omega}_i$ and angular velocity $^{i}\omega_i$ of link frame $\{i\}$ and $\ddot{\theta}_{i+1}$. Also, note that for a prismatic joint,

$$\ddot{\theta}_{i+1} = \dot{\theta}_{i+1} = 0 \quad (80)$$

so that (78) and (79) simplify (for prismatic joints) to:

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i \quad (81)$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i \quad (82)$$

C. Linear accelerations of link frame origins

In Section 2, we derived the following relationship for the propagation of linear accelerations:

$$^{A}\dot{V}_Q = {}^{A}\dot{V}_{BORG} + {}^{A}_{B}R({}^{B}\dot{V}_Q) + 2\,{}^{A}\Omega_B \times [{}^{A}_{B}R({}^{B}V_Q)] + {}^{A}\dot{\Omega}_B \times [{}^{A}_{B}R({}^{B}Q)] + {}^{A}\Omega_B \times [{}^{A}\Omega_B \times {}^{A}_{B}R({}^{B}Q)] \quad (83)$$

We will now convert equation (83) into link-specific form as we did above for angular accelerations. First, let us make the following substitutions:

$$A = U \quad (84)$$

$$B = i \quad (85)$$

$$Q = i+1, ORG \text{ (for subscripts)} \quad (86)$$

Given these substitutions, note that ^{B}Q now denotes the origin of link frame $\{i+1\}$ in terms of link frame $\{i\}$; in short,

$$^{B}Q = {}^{i}P_{i+1,ORG} \quad (87)$$

Thus, equation (83) becomes,

$$\begin{aligned} ^{U}\dot{V}_{i+1,ORG} = {} & {}^{U}\dot{V}_{i,ORG} + {}^{U}_{i}R({}^{i}\dot{V}_{i+1,ORG}) + 2\,{}^{U}\Omega_i \times [{}^{U}_{i}R({}^{i}V_{i+1,ORG})] + \\ & {}^{U}\dot{\Omega}_i \times [{}^{U}_{i}R({}^{i}P_{i+1,ORG})] + {}^{U}\Omega_i \times [{}^{U}\Omega_i \times {}^{U}_{i}R({}^{i}P_{i+1,ORG})] \end{aligned} \quad (88)$$

Let us now pre-multiply equation (88) by $^{i+1}_{U}R$:

$$\begin{aligned} ^{i+1}_{U}R\,{}^{U}\dot{V}_{i+1,ORG} = {} & {}^{i+1}_{U}R\,{}^{U}\dot{V}_{i,ORG} + {}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}\dot{V}_{i+1,ORG}) + \\ & 2\,{}^{i+1}_{U}R\,{}^{U}\Omega_i \times [{}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}V_{i+1,ORG})] + \\ & {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_i \times [{}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}P_{i+1,ORG})] + \\ & {}^{i+1}_{U}R\,{}^{U}\Omega_i \times [{}^{i+1}_{U}R\,{}^{U}\Omega_i \times {}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}P_{i+1,ORG})] \end{aligned} \quad (89)$$

Note that we used vector identity (65) in distributing $^{i+1}_{U}R$ over the cross product of vectors in (89). Let us now consider each of the terms in equation (89) one by one. The left-hand side of (89) can be written in short-hand notation as:

$$^{i+1}_{U}R\,{}^{U}\dot{V}_{i+1,ORG} = {}^{i+1}\dot{v}_{i+1} \text{ (by definition)} \quad (90)$$

Similarly for the first term on the right-hand side of equation (89):

$$ {}^{i+1}_{U}R\,{}^{U}\dot{V}_{i,ORG} = {}^{i+1}_{i}R({}^{i}_{U}R\,{}^{U}\dot{V}_{i,ORG}) = {}^{i+1}_{i}R\,{}^{i}\dot{v}_{i} \text{ (by definition)} \tag{91}$$

Equations (90) and (91) simplify equation (89) to:

$$\begin{aligned} {}^{i+1}\dot{v}_{i+1} = {} & {}^{i+1}_{i}R\,{}^{i}\dot{v}_{i} + {}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}\dot{V}_{i+1,ORG}) + \\ & 2\,{}^{i+1}_{U}R\,{}^{U}\Omega_{i}\times[{}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}V_{i+1,ORG})] + \\ & {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_{i}\times[{}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}P_{i+1,ORG})] + \\ & {}^{i+1}_{U}R\,{}^{U}\Omega_{i}\times[{}^{i+1}_{U}R\,{}^{U}\Omega_{i}\times{}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}P_{i+1,ORG})] \end{aligned} \tag{92}$$

Let us now consider the second term on the right-hand side of equation (92). The notation ${}^{i}\dot{V}_{i+1,ORG}$ indicates the linear acceleration of the origin of link frame $\{i+1\}$ with respect to (and in terms of) link frame $\{i\}$. Thus, for a serial-link manipulator, we can write:

$$ {}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}\dot{V}_{i+1,ORG}) = {}^{i+1}_{i}R({}^{i}\dot{V}_{i+1,ORG}) = \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \tag{93}$$

Similarly,

$$ {}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}V_{i+1,ORG}) = {}^{i+1}_{i}R({}^{i}V_{i+1,ORG}) = \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \tag{94}$$

Note that linear relationships in (93) and (94) are analogous to the manipulator-specific angular relationships in (73) and (74). Equation (92) now simplifies to:

$$\begin{aligned} {}^{i+1}\dot{v}_{i+1} = {} & {}^{i+1}_{i}R\,{}^{i}\dot{v}_{i} + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \\ & 2\,{}^{i+1}_{U}R\,{}^{U}\Omega_{i}\times\dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \\ & {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_{i}\times[{}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}P_{i+1,ORG})] + \\ & {}^{i+1}_{U}R\,{}^{U}\Omega_{i}\times[{}^{i+1}_{U}R\,{}^{U}\Omega_{i}\times{}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}P_{i+1,ORG})] \end{aligned} \tag{95}$$

Let us now consider the angular velocity and acceleration terms in (95). From prior discussion,

$$ {}^{i+1}_{U}R\,{}^{U}\Omega_{i} = {}^{i+1}_{i}R({}^{i}_{U}R\,{}^{U}\Omega_{i}) = {}^{i+1}_{i}R\,{}^{i}\omega_{i} \tag{96}$$

$$ {}^{i+1}_{U}R\,{}^{U}\dot{\Omega}_{i} = {}^{i+1}_{i}R({}^{i}_{U}R\,{}^{U}\dot{\Omega}_{i}) = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_{i} \tag{97}$$

so that equation (95) further reduces to:

$$\begin{aligned} {}^{i+1}\dot{v}_{i+1} = {} & {}^{i+1}_{i}R\,{}^{i}\dot{v}_{i} + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \\ & 2({}^{i+1}_{i}R\,{}^{i}\omega_{i})\times\dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \\ & {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_{i}\times[{}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}P_{i+1,ORG})] + \\ & {}^{i+1}_{i}R\,{}^{i}\omega_{i}\times[{}^{i+1}_{i}R\,{}^{i}\omega_{i}\times{}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}P_{i+1,ORG})] \end{aligned} \tag{98}$$

Next, let us make the following substitution in (98):

$$ {}^{i+1}_{U}R\,{}^{U}_{i}R({}^{i}P_{i+1,ORG}) = {}^{i+1}_{i}R({}^{i}P_{i+1,ORG}) = {}^{i+1}_{i}R({}^{i}P_{i+1}) \tag{99}$$

where ${}^{i}P_{i+1}$ is simply short-hand notation for ${}^{i}P_{i+1,ORG}$. Thus, equation (98) reduces to:

$$\begin{aligned} {}^{i+1}\dot{v}_{i+1} = {} & {}^{i+1}_{i}R\,{}^{i}\dot{v}_i + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + 2({}^{i+1}_{i}R\,{}^{i}\omega_i)\times\dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \\ & {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i\times[{}^{i+1}_{i}R({}^{i}P_{i+1})] + {}^{i+1}_{i}R\,{}^{i}\omega_i\times[{}^{i+1}_{i}R\,{}^{i}\omega_i\times{}^{i+1}_{i}R({}^{i}P_{i+1})] \end{aligned} \tag{100}$$

Finally, note that the relationship in (65) can be rewritten as,

$$(RQ)\times(RP) = R(Q\times P) \tag{101}$$

so that we can rearrange and group terms in equation (100):

$$\begin{aligned} {}^{i+1}\dot{v}_{i+1} = {} & {}^{i+1}_{i}R\,{}^{i}\dot{v}_i + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + 2({}^{i+1}_{i}R\,{}^{i}\omega_i)\times\dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \\ & {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i\times[{}^{i+1}_{i}R({}^{i}P_{i+1})] + {}^{i+1}_{i}R\,{}^{i}\omega_i\times[{}^{i+1}_{i}R\,{}^{i}\omega_i\times{}^{i+1}_{i}R({}^{i}P_{i+1})] \end{aligned} \tag{102}$$

$$\begin{aligned} {}^{i+1}\dot{v}_{i+1} = {} & {}^{i+1}_{i}R\,{}^{i}\dot{v}_i + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + 2({}^{i+1}_{i}R\,{}^{i}\omega_i)\times\dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \\ & {}^{i+1}_{i}R({}^{i}\dot{\omega}_i\times{}^{i}P_{i+1}) + {}^{i+1}_{i}R[{}^{i}\omega_i\times({}^{i}\omega_i\times{}^{i}P_{i+1})] \end{aligned} \tag{103}$$

Thus (for a prismatic joints):

$$\begin{aligned} {}^{i+1}\dot{v}_{i+1} = {} & {}^{i+1}_{i}R[{}^{i}\dot{\omega}_i\times{}^{i}P_{i+1} + {}^{i}\omega_i\times({}^{i}\omega_i\times{}^{i}P_{i+1}) + {}^{i}\dot{v}_i] + \\ & 2({}^{i+1}_{i}R\,{}^{i}\omega_i)\times\dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \end{aligned} \tag{104}$$

Note that equation (104) allow us to compute the linear acceleration ${}^{i+1}\dot{v}_{i+1}$ of link frame $\{i+1\}$ in terms of the angular acceleration ${}^{i}\dot{\omega}_i$, linear acceleration ${}^{i}\dot{v}_i$ and angular velocity ${}^{i}\omega_i$ of link frame $\{i\}$, and $\dot{d}_{i+1}$ and $\ddot{d}_{i+1}$. Also, note that for a revolute joint,

$$\ddot{d}_{i+1} = \dot{d}_{i+1} = 0 \tag{105}$$

so that (104) simplifies (for revolute joints) to:

$${}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R[{}^{i}\dot{\omega}_i\times{}^{i}P_{i+1} + {}^{i}\omega_i\times({}^{i}\omega_i\times{}^{i}P_{i+1}) + {}^{i}\dot{v}_i]. \tag{106}$$

D. Linear acceleration of a link's center of mass

In order to apply Newton's second law of motion in equation (33), we need to know not just the linear acceleration of link frame $\{i\}$ but of the center of mass C_i of link i as well. Once again, we will begin with the relationship for the propagation of linear accelerations derived in Section 2:

$${}^{A}\dot{V}_Q = {}^{A}\dot{V}_{BORG} + {}^{A}_{B}R({}^{B}\dot{V}_Q) + 2\,{}^{A}\Omega_B\times[{}^{A}_{B}R({}^{B}V_Q)] + {}^{A}\dot{\Omega}_B\times[{}^{A}_{B}R({}^{B}Q)] + {}^{A}\Omega_B\times[{}^{A}\Omega_B\times{}^{A}_{B}R({}^{B}Q)] \tag{107}$$

Let us make the following substitutions:

$$A = U \tag{108}$$

$$B = i \tag{109}$$

$$Q = C_i \text{ (for subscripts)} \tag{110}$$

Given these substitutions, note that ${}^{B}Q$ now denotes the origin of the center of mass of link frame $\{i+1\}$ in terms of link frame $\{i\}$; in short,

$${}^{B}Q = {}^{i}P_{C_i} \tag{111}$$

Thus, equation (107) becomes,

$$
\begin{aligned}
{}^{U}\dot{V}_{C_i} &= {}^{U}\dot{V}_{i,\,ORG} + {}^{U}_{i}R({}^{i}\dot{V}_{C_i}) + 2\,{}^{U}\Omega_i \times [{}^{U}_{i}R({}^{i}V_{C_i})] + \\
&\quad {}^{U}\dot{\Omega}_i \times [{}^{U}_{i}R({}^{i}P_{C_i})] + {}^{U}\Omega_i \times [{}^{U}\Omega_i \times {}^{U}_{i}R({}^{i}P_{C_i})]
\end{aligned} \tag{112}
$$

In equation (112), note that for rigid links,

$$
{}^{i}\dot{V}_{C_i} = {}^{i}V_{C_i} = 0 \tag{113}
$$

so that (112) reduces from,

$$
\begin{aligned}
{}^{U}\dot{V}_{C_i} &= {}^{U}\dot{V}_{i,\,ORG} + {}^{U}_{i}R({}^{i}\dot{V}_{C_i}) + 2\,{}^{U}\Omega_i \times [{}^{U}_{i}R({}^{i}V_{C_i})] + \\
&\quad {}^{U}\dot{\Omega}_i \times [{}^{U}_{i}R({}^{i}P_{C_i})] + {}^{U}\Omega_i \times [{}^{U}\Omega_i \times {}^{U}_{i}R({}^{i}P_{C_i})]
\end{aligned} \tag{114}
$$

to the simplified form,

$$
{}^{U}\dot{V}_{C_i} = {}^{U}\dot{V}_{i,\,ORG} + {}^{U}\dot{\Omega}_i \times [{}^{U}_{i}R({}^{i}P_{C_i})] + {}^{U}\Omega_i \times [{}^{U}\Omega_i \times {}^{U}_{i}R({}^{i}P_{C_i})] \tag{115}
$$

Let us now pre-multiply equation (115) by ${}^{i}_{U}R$:

$$
{}^{i}_{U}R\,{}^{U}\dot{V}_{C_i} = {}^{i}_{U}R\,{}^{U}\dot{V}_{i,\,ORG} + {}^{i}_{U}R\,{}^{U}\dot{\Omega}_i \times [{}^{i}_{U}R\,{}^{U}_{i}R({}^{i}P_{C_i})] + {}^{i}_{U}R\,{}^{U}\Omega_i \times [{}^{i}_{U}R\,{}^{U}\Omega_i \times {}^{i}_{U}R\,{}^{U}_{i}R({}^{i}P_{C_i})] \tag{116}
$$

Note that we used vector identity (65) in distributing ${}^{i}_{U}R$ over the cross product of vectors in (116). Similar to earlier derivation, we now make the following substitutions:

$$
{}^{i}_{U}R\,{}^{U}\dot{V}_{C_i} = {}^{i}\dot{v}_{C_i} \text{ (by definition)} \tag{117}
$$

$$
{}^{i}_{U}R\,{}^{U}\dot{V}_{i,\,ORG} = {}^{i}\dot{v}_i \text{ (by definition)} \tag{118}
$$

$$
{}^{i}_{U}R\,{}^{U}\Omega_i = {}^{i}\omega_i \text{ and } {}^{i}_{U}R\,{}^{U}\dot{\Omega}_i = {}^{i}\dot{\omega}_i \text{ (by definition)} \tag{119}
$$

Equation (116) consequently reduces to:

$$
{}^{i}\dot{v}_{C_i} = {}^{i}\dot{v}_i + {}^{i}\dot{\omega}_i \times [{}^{i}_{U}R\,{}^{U}_{i}R({}^{i}P_{C_i})] + {}^{i}\omega_i \times [{}^{i}\omega_i \times {}^{i}_{U}R\,{}^{U}_{i}R({}^{i}P_{C_i})] \tag{120}
$$

Noting that,

$$
{}^{i}_{U}R\,{}^{U}_{i}R({}^{i}P_{C_i}) = {}^{i}P_{C_i} \tag{121}
$$

equation (120) now reduces to:

$$
{}^{i}\dot{v}_{C_i} = {}^{i}\dot{v}_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times [{}^{i}\omega_i \times {}^{i}P_{C_i}] \tag{122}
$$

$$
{}^{i}\dot{v}_{C_i} = {}^{i}\dot{v}_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times [{}^{i}\omega_i \times {}^{i}P_{C_i}] \tag{123}
$$

5. Link-specific equations of motion

A. Manipulator-specific equations of motions

We can rewrite the basic equations of motion in (33) and (37) in more link specific notation. Specifically,

$$F_i = m_i \dot{v}_{C_i} \tag{124}$$

$$N_i = {}^{C_i}I_i \dot{\omega}_i + \omega_i \times {}^{C_i}I_i \omega_i \tag{125}$$

where,

$$F_i = \text{net force acting on link } i, \tag{126}$$

$$m_i = \text{mass of link } i, \tag{127}$$

$$\dot{v}_{C_i} = \text{acceleration of center of mass of link } i, \tag{128}$$

$$N_i = \text{net moment acting on link } i, \tag{129}$$

$${}^{C_i}I_i = \text{inertia tensor, written in frame } \{C_i\} \text{ located at the center of mass of link } i, \tag{130}$$

$$\omega_i = \text{angular velocity of link } i \text{, and,} \tag{131}$$

$$\dot{\omega}_i = \text{angular acceleration of link } i. \tag{132}$$

We can now completely and succinctly write the outward propagation of velocities and net forces/moments. In the following subsections, we do so for revolute and prismatic joints, respectively.

B. Summary of outward iteration

1. Revolute joints:

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \tag{133}$$

$${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + ({}^{i+1}_{i}R\,{}^{i}\omega_i) \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \tag{134}$$

$${}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R[{}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i] \tag{135}$$

2. Prismatic joints:

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i \tag{136}$$

$${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i \tag{137}$$

$$\begin{aligned}{}^{i+1}\dot{v}_{i+1} = {}& {}^{i+1}_{i}R[{}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i] + \\ & 2({}^{i+1}_{i}R\,{}^{i}\omega_i) \times \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}\end{aligned} \tag{138}$$

3. Both joint types:

$${}^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{v}_{i+1} + {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times [{}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}] \tag{139}$$

$$ {}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}} \tag{140} $$

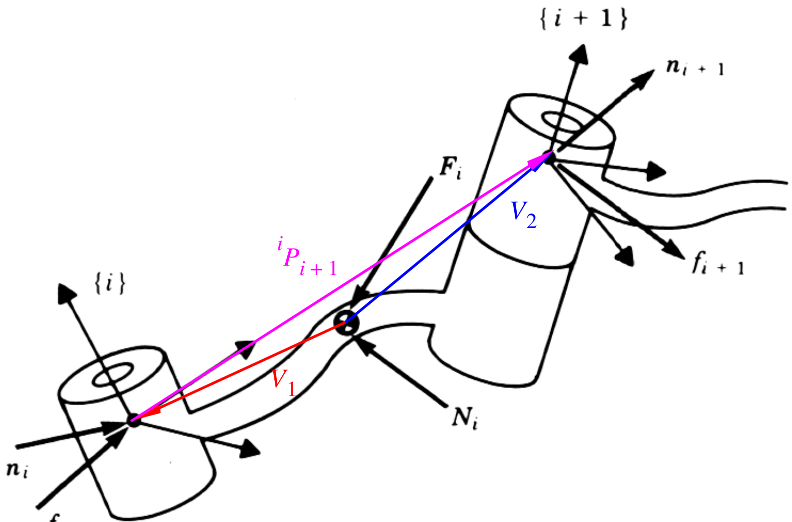
$$ {}^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1} \tag{141} $$

6. Force/moment balance equations

A. Introduction

Equations (133) through (141) give us the net forces/moments at a given link required to cause the desired/known joint motion $(\Theta, \dot{\Theta}, \ddot{\Theta})$. We now must figure out what part of those net forces/moments must be supplied by the joint actuators. To do this, we will write force/moment balance equations about the center of mass of each link.

Consider Figure 3 below, which illustrates the forces and moments acting on link i. In Figure 3 we use the following notation:



(Craig, Fig. 6.5)

Figure 3: Forces and moments action on link *i*.

f_i = force exerted on link i by link $i-1$, (142)

f_{i+1} = force exerted on link $i+1$ by link i, (143)

n_i = moment exerted on link i by link $i-1$, (144)

n_{i+1} = moment exerted on link $i+1$ by link i, (145)

and, as before,

F_i = net force acting on link i, and, (146)

N_i = net moment acting on link i. (147)

Also,

${}^iP_{i+1}$ = vector from the origin of frame $\{i\}$ to the origin of frame $\{i+1\}$, (148)

V_1 = vector from the center of mass of link i to the origin of coordinate frame $\{i\}$, and, (149)

$$V_2 \; = \text{vector from the center of mass of link } i \text{ to the origin of coordinate frame } \{i+1\}. \tag{150}$$

B. Force balance equation

Given the above notation, we can write the force balance equation for link i:

$$F_i = f_i - f_{i+1} \tag{151}$$

We can, of course, express equation (151) with respect to any coordinate frame. Let us rewrite (151) in terms of coordinate frame $\{i\}$:

$$^iF_i = {}^if_i - {}^if_{i+1} \tag{152}$$

$$^iF_i = {}^if_i - ({}_{i+1}^{\;\;i}R)^{i+1}f_{i+1} \tag{153}$$

$$^iF_i = {}^if_i - ({}_{i+1}^{\;\;i}R)^{i+1}f_{i+1} \tag{154}$$

C. Moment balance equation

Now, let us write the moment balance equation about the center of mass of link i:

$$N_i = n_i - n_{i+1} + V_1 \times f_i - V_2 \times f_{i+1} \tag{155}$$

Note that $V_1 \times f_i$ and $-V_2 \times f_{i+1}$ give the moments induced by forces f_i and $-f_{i+1}$, respectively, about the center of mass of link $\{i\}$. Let us now write expressions for V_1 and V_2 in link-specific notation:

$$V_1 = (-{}^iP_{C_i}) \tag{156}$$

$$V_2 = ({}^iP_{i+1} - {}^iP_{C_i}). \tag{157}$$

Substituting (156) and (157) into (155) and expressing with respect to frame $\{i\}$:

$$^iN_i = {}^in_i - {}^in_{i+1} + (-{}^iP_{C_i}) \times {}^if_i - ({}^iP_{i+1} - {}^iP_{C_i}) \times {}^if_{i+1} \tag{158}$$

Rearranging terms and keeping equation (152) in mind,

$$^iN_i = {}^in_i - {}^in_{i+1} - {}^iP_{C_i} \times ({}^if_i - {}^if_{i+1}) - {}^iP_{i+1} \times {}^if_{i+1} \tag{159}$$

$$^iN_i = {}^in_i - {}^in_{i+1} - {}^iP_{C_i} \times {}^iF_i - {}^iP_{i+1} \times {}^if_{i+1} \tag{160}$$

$$^iN_i = {}^in_i - {}^in_{i+1} - {}^iP_{C_i} \times {}^iF_i - {}^iP_{i+1} \times [({}_{i+1}^{\;\;i}R)^{i+1}f_{i+1}] \tag{161}$$

$$^iN_i = {}^in_i - {}^in_{i+1} - {}^iP_{C_i} \times {}^iF_i - {}^iP_{i+1} \times [({}_{i+1}^{\;\;i}R)^{i+1}f_{i+1}] \tag{162}$$

D. Inward iteration of link forces and moments

Thus the force and moment balance equations at link i are given by,

$$^iF_i = {}^if_i - ({}_{i+1}^{\;\;i}R)^{i+1}f_{i+1} \text{ and} \tag{163}$$

$$^iN_i = {}^in_i - {}^in_{i+1} - {}^iP_{C_i} \times {}^iF_i - {}^iP_{i+1} \times [({}_{i+1}^{\;\;i}R)^{i+1}f_{i+1}]. \tag{164}$$

We can rewrite equations (163) and (164) as iterations that propagate ${}^{i}f_i$ and ${}^{i}n_i$ from the end-effector to the link frame $\{1\}$:

$$ {}^{i}f_i = ({}_{i+1}^{i}R)^{i+1}f_{i+1} + {}^{i}F_i \tag{165} $$

$$ {}^{i}n_i = {}^{i}N_i + ({}_{i+1}^{i}R)^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times [({}_{i+1}^{i}R)^{i+1}f_{i+1}] \tag{166} $$

Note that equations (165) and (166) allow us to recursively compute the forces and moments that each link exerts on its neighboring links by inwardly propagating from coordinate frame $\{N\}$ to frame $\{1\}$. The last remaining question is, once equations (165) and (166) are computed, what should be the torques/forces for the actuators to achieve the desired joint motion? All components of the force and moment vectors ${}^{i}f_i$ and ${}^{i}n_i$ are resisted by the structure of the mechanism itself, except for the torque/force about/along the joint axis. Therefore the required *torque* for a revolute joint i is given by,

$$ \tau_i = {}^{i}n_i \cdot {}^{i}\hat{Z}_i \tag{167} $$

while the required *force* for a prismatic joint i is given by,

$$ \tau_i = {}^{i}f_i \cdot {}^{i}\hat{Z}_i . \tag{168} $$

7. Complete formulation of the iterative Newton-Euler dynamics

This section summarizes the complete formulation of the iterative Newton-Euler dynamics model. It consists of (1) the outward propagation of angular velocities and linear and angular accelerations, (2) the outward propagation of net moments and forces acting on the links, and (3) the inward propagation of forces and torques between links. Collectively, equations (170) through (182) implicitly define the relationship we were looking for at the beginning of this discussion — namely,

$$ \tau = h(\Theta, \dot{\Theta}, \ddot{\Theta}) \tag{169} $$

A. Outward iteration

1. Revolute joints:

$$ {}^{i+1}\omega_{i+1} = {}_{i}^{i+1}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \tag{170} $$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}_{i}^{i+1}R\,{}^{i}\dot{\omega}_i + ({}_{i}^{i+1}R\,{}^{i}\omega_i) \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \tag{171} $$

$$ {}^{i+1}\dot{v}_{i+1} = {}_{i}^{i+1}R[{}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i] \tag{172} $$

2. Prismatic joints:

$$ {}^{i+1}\omega_{i+1} = {}_{i}^{i+1}R\,{}^{i}\omega_i \tag{173} $$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}_{i}^{i+1}R\,{}^{i}\dot{\omega}_i \tag{174} $$

$$ {}^{i+1}\dot{v}_{i+1} = {}_{i}^{i+1}R[{}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i] + 2({}_{i}^{i+1}R\,{}^{i}\omega_i) \times \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \tag{175} $$

3. Both joint types:

$$ {}^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{v}_{i+1} + {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times [{}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}] \tag{176} $$

$$^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}} \tag{177}$$

$$^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1} \tag{178}$$

B. Inward iteration

1. Both joint types:

$$^{i}f_i = ({}_{i+1}^{i}R)\,{}^{i+1}f_{i+1} + {}^{i}F_i \tag{179}$$

$$^{i}n_i = {}^{i}N_i + ({}_{i+1}^{i}R)\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times [({}_{i+1}^{i}R)\,{}^{i+1}f_{i+1}] \tag{180}$$

2. Revolute joints:

$$\tau_i = {}^{i}n_i \cdot {}^{i}\hat{Z}_i \tag{181}$$

3. Prismatic joints:

$$\tau_i = {}^{i}f_i \cdot {}^{i}\hat{Z}_i . \tag{182}$$

C. Initialization of propagations

In order to compute equations (170) through (175), we need to know ${}^{0}\omega_0$, ${}^{0}\dot{\omega}_0$ and ${}^{0}\dot{v}_0$ —that is, the angular velocity, and linear and angular acceleration of the base coordinate frame $\{0\}$. For a fixed-base manipulator,

$$^{0}\omega_0 = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^T , \tag{183}$$

$$^{0}\dot{\omega}_0 = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^T \text{, and,} \tag{184}$$

$$^{0}\dot{v}_0 = -{}^{0}G , \tag{185}$$

where ${}^{0}G$ denotes the gravity vector. Note that (185) is equivalent to saying that the base of the robot is accelerating upward with acceleration g, and therefore easily incorporates the effects of gravity loading on the links without any additional effort.

In order to compute equations (179) and (180), we need to know ${}^{N+1}f_{N+1}$ and ${}^{N+1}n_{N+1}$ — that is, the forces and moments from the environment acting on the end-effector of the manipulator. When the manipulator end-effector is not in contact with any object or obstacle, these are simply given by,

$$^{N+1}f_{N+1} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^T \text{, and,} \tag{186}$$

$$^{N+1}n_{N+1} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^T . \tag{187}$$