C4 Computer Vision

4 Lectures — Michaelmas Term 2003
1 Tutorial Sheet — Prof A. Zisserman

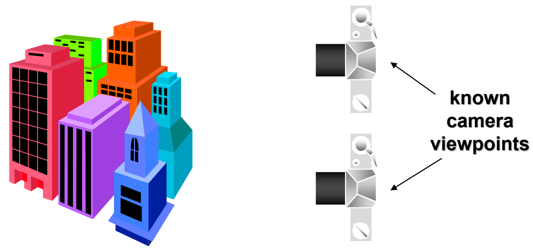
Overview

- Lecture 1: Stereo Reconstruction I: epipolar geometry, fundamental matrix.
- Lecture 2: Stereo Reconstruction II: correspondence algorithms, triangulation.
- Lecture 3: Structure and Motion: ambiguities, computing the fundamental matrix, recovering ego-motion, applications.
- Lecture 4: Object detection: the adaBoost algorithm for face detection.

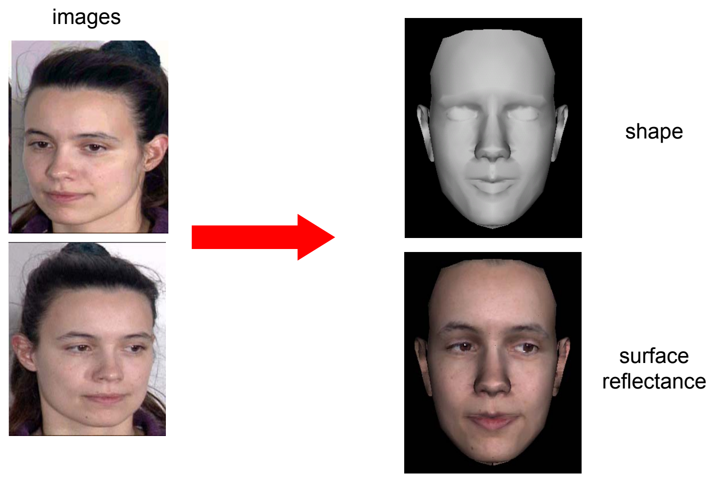
Further reading (www addresses) and the lecture notes are on http://www.robots.ox.ac.uk/~az/lectures

Stereo Reconstruction

Shape (3D) from two (or more) images



Example



Scenarios

The two images can arise from

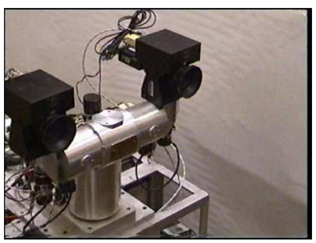
- A stereo rig consisting of two cameras
  - the two images are acquired simultaneously

or

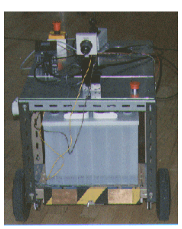
- A single moving camera (static scene)
  - the two images are acquired sequentially

The two scenarios are geometrically equivalent

Stereo head



Camera on a mobile vehicle



The objective

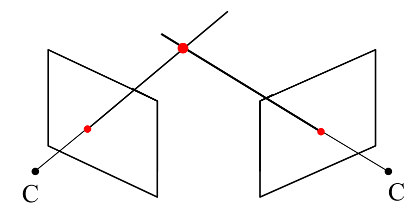
Given two images of a scene acquired by known cameras compute the 3D position of the scene (structure recovery)

Basic principle: triangulate from corresponding image points

- Determine 3D point at intersection of two back-projected rays

Corresponding points are images of the same scene point

Triangulation



The back-projected points generate rays which intersect at the 3D scene point

An algorithm for stereo reconstruction

1. For each point in the first image determine the corresponding point in the second image

   (this is a search problem)

2. For each pair of matched points determine the 3D point by triangulation

   (this is an estimation problem)

The correspondence problem

Given a point x in one image find the corresponding point in the other image

This appears to be a 2D search problem, but it is reduced to a 1D search by the epipolar constraint

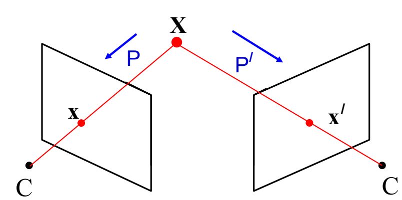
Outline

1. Epipolar geometry
   - the geometry of two cameras
   - reduces the correspondence problem to a line search
2. Stereo correspondence algorithms
3. Triangulation

Notation

The two cameras are P and P$'$, and a 3D point $\mathbf{X}$ is imaged as

$$\mathbf{x} = \mathrm{P}\mathbf{X} \qquad \mathbf{x}' = \mathrm{P}'\mathbf{X}$$



P : 3×4 matrix

$\mathbf{X}$: 4-vector

$\mathbf{x}$: 3-vector

Warning

for equations involving homogeneous quantities '=' means 'equal up to scale'

Epipolar geometry

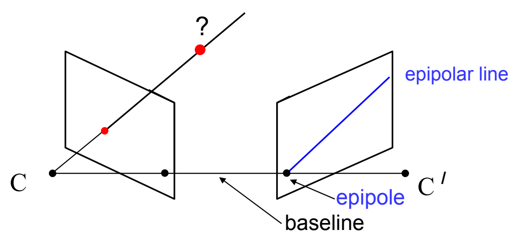
Epipolar line

Epipolar constraint

- Reduces correspondence problem to 1D search along an epipolar line

Epipolar geometry

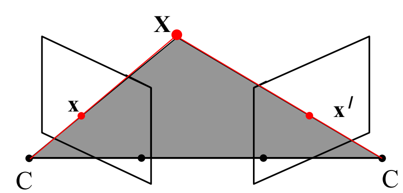
Given an image point in one view, where is the corresponding point in the other view?



- A point in one view "generates" an epipolar line in the other view
- The corresponding point lies on this line

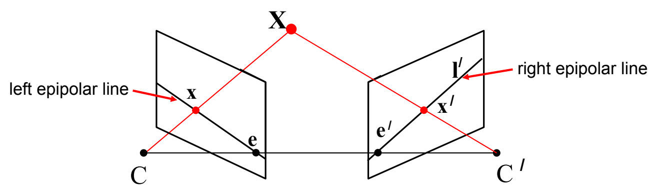
Epipolar geometry continued

Epipolar geometry is a consequence of the coplanarity of the camera centres and scene point



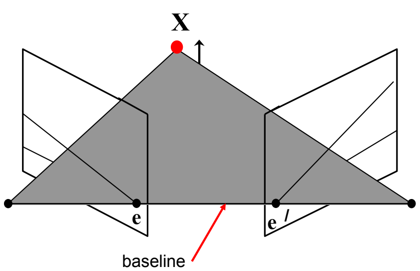
The camera centres, corresponding points and scene point lie in a single plane, known as the epipolar plane

Nomenclature



- The epipolar line $\mathbf{l}'$ is the image of the ray through $\mathbf{x}$
- The epipole $\mathbf{e}$ is the point of intersection of the line joining the camera centres with the image plane
  - this line is the baseline for a stereo rig, and
  - the translation vector for a moving camera
- The epipole is the image of the centre of the other camera: $\mathbf{e} = \mathrm{P}\mathbf{C}'$, $\mathbf{e}' = \mathrm{P}'\mathbf{C}$

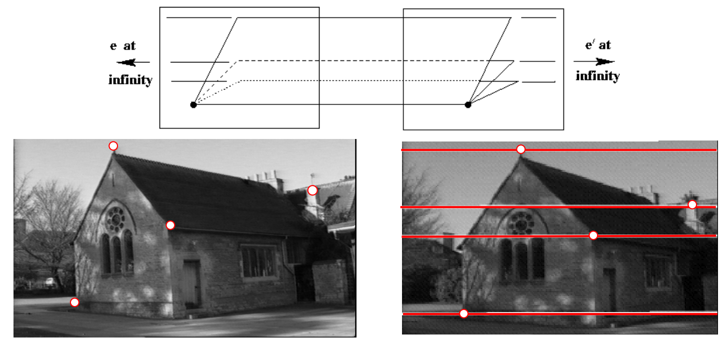
The epipolar pencil



As the position of the 3D point $\mathbf{X}$ varies, the epipolar planes "rotate" about the baseline. This family of planes is known as an epipolar pencil. All epipolar lines intersect at the epipole.

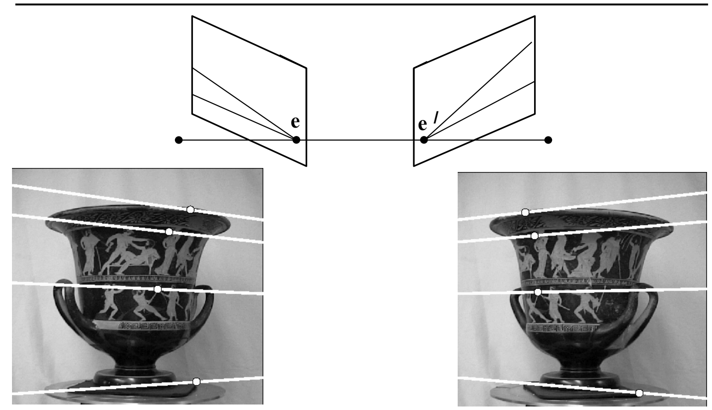
(a pencil is a one parameter family)

Epipolar geometry example I: parallel cameras



Epipolar geometry depends only on the relative pose (position and orientation) and internal parameters of the two cameras, i.e. the position of the camera centres and image planes. It does not depend on the scene structure (3D points external to the camera).

Epipolar geometry example II: converging cameras



Note, epipolar lines are in general not parallel

Homogeneous notation for lines

Recall that a point (x, y) in 2D is represented by the homogeneous 3-vector $\mathbf{x} = (x_1, x_2, x_3)^\top$, where $x = x_1/x_3, y = x_2/x_3$

A line in 2D is represented by the homogeneous 3-vector

$$\mathbf{l} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$$

which is the line $l_1 x + l_2 y + l_3 = 0$.

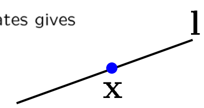
Example represent the line $y = 1$ as a homogeneous vector.

Write the line as $-y + 1 = 0$ then $l_1 = 0, l_2 = -1, l_3 = 1$, and $\mathbf{l} = (0, -1, 1)^\top$.

Note that $\mu(l_1 x + l_2 y + l_3) = 0$ represents the same line (only the ratio of the homogeneous line coordinates is significant).

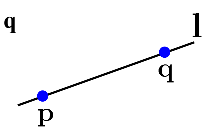
Writing both the point and line in homogeneous coordinates gives

$$l_1 x_1 + l_2 x_2 + l_3 x_3 = 0$$



- point on line $\mathbf{l}.\mathbf{x} = 0$ or $\mathbf{l}^\top \mathbf{x} = 0$ or $\mathbf{x}^\top \mathbf{l} = 0$

- The line $\mathbf{l}$ through the two points $\mathbf{p}$ and $\mathbf{q}$ is $\mathbf{l} = \mathbf{p} \times \mathbf{q}$



Proof

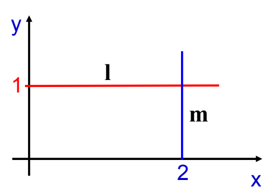
$$\mathbf{l}.\mathbf{p} = (\mathbf{p} \times \mathbf{q}).\mathbf{p} = 0 \qquad \mathbf{l}.\mathbf{q} = (\mathbf{p} \times \mathbf{q}).\mathbf{q} = 0$$

- The intersection of two lines $\mathbf{l}$ and $\mathbf{m}$ is the point $\mathbf{x} = \mathbf{l} \times \mathbf{m}$

Example: compute the point of intersection of the two lines $\mathbf{l}$ and $\mathbf{m}$ in the figure below

$$\mathbf{l} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{m} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{m} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$$



which is the point (2,1)

Matrix representation of the vector cross product

The vector product $\mathbf{v} \times \mathbf{x}$ can be represented as a matrix multiplication

$$\mathbf{v} \times \mathbf{x} = \begin{pmatrix} v_2 x_3 - v_3 x_2 \\ v_3 x_1 - v_1 x_3 \\ v_1 x_2 - v_2 x_1 \end{pmatrix} = [\mathbf{v}]_\times \mathbf{x}$$

where

$$[\mathbf{v}]_\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

- $[\mathbf{v}]_\times$ is a 3×3 skew-symmetric matrix of rank 2.
- $\mathbf{v}$ is the null-vector of $[\mathbf{v}]_\times$, since $\mathbf{v} \times \mathbf{v} = [\mathbf{v}]_\times \mathbf{v} = \mathbf{0}$.

Example: compute the cross product of $\mathbf{l}$ and $\mathbf{m}$

$$\mathbf{l} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{m} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \qquad [\mathbf{v}]_\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{m} = [\mathbf{l}]_\times \mathbf{m} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$$

Note

$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Algebraic representation of epipolar geometry

We know that the epipolar geometry defines a mapping

$$\mathbf{x} \rightarrow \mathbf{l}'$$

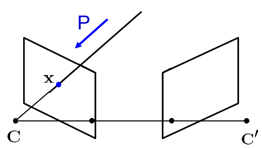
point in first image → epipolar line in second image

- the map ony depends on the cameras $\mathtt{P}, \mathtt{P}'$ (not on structure)
- it will be shown that the map is linear and can be written as $\mathbf{l}' = \mathtt{F}\mathbf{x}$, where $\mathtt{F}$ is a 3×3 matrix called the fundamental matrix

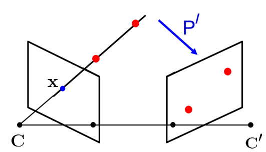
Derivation of the algebraic expression $\mathbf{l}' = \mathtt{F}\mathbf{x}$

Outline

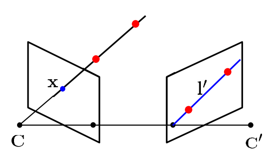
Step 1: for a point x in the first image back project a ray with camera P



Step 2: choose two points on the ray and project into the second image with camera P′



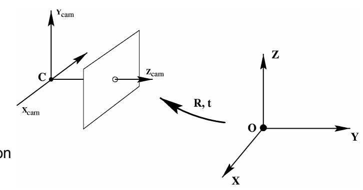
Step 3: compute the line through the two image points using the relation $\mathbf{l}' = \mathbf{p} \times \mathbf{q}$



- choose camera matrices

$$\mathtt{P} = \mathtt{K}\,[\,\mathtt{R} \mid \mathbf{t}\,]$$

internal calibration; rotation; translation — from world to camera coordinate frame

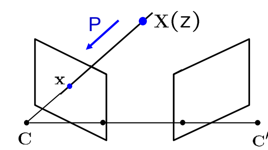


- first camera $\mathtt{P} = \mathtt{K}\,[\,\mathtt{I} \mid \mathbf{0}\,]$

world coordinate frame aligned with first camera

- second camera $\mathtt{P}' = \mathtt{K}'\,[\,\mathtt{R} \mid \mathbf{t}\,]$

Step 1: for a point x in the first image back project a ray with camera $\mathtt{P} = \mathtt{K}\,[\,\mathtt{I} \mid \mathbf{0}\,]$



A point $\mathbf{x}$ back projects to a ray

$$\begin{pmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{pmatrix} = \mathrm{Z}\mathtt{K}^{-1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \mathrm{Z}\mathtt{K}^{-1}\mathbf{x}$$

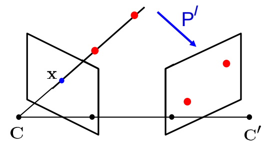
where $\mathbf{Z}$ is the point's depth, since

$$\mathbf{X}(\mathrm{Z}) = \begin{pmatrix} \mathrm{Z}\mathtt{K}^{-1}\mathbf{x} \\ 1 \end{pmatrix}$$

satisfies

$$\mathtt{P}\mathbf{X}(\mathrm{Z}) = \mathtt{K}[\mathtt{I} \mid \mathbf{0}]\mathbf{X}(\mathrm{Z}) = \mathbf{x}$$

Step 2: choose two points on the ray and project into the second image with camera P′



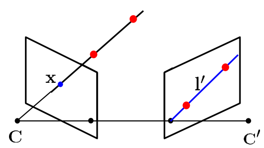
Consider two points on the ray $\mathtt{X}(\mathtt{z}) = \begin{pmatrix} \mathtt{z}\mathtt{K}^{-1}\mathbf{x} \\ 1 \end{pmatrix}$

- $\mathbf{Z}$ = 0 is the camera centre $\begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$
- $\mathbf{Z} = \infty$ is the point at infinity $\begin{pmatrix} \mathtt{K}^{-1}\mathbf{x} \\ 0 \end{pmatrix}$

Project these two points into the second view

$$\mathtt{P}' \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} = \mathtt{K}'[\mathtt{R} \mid \mathbf{t}] \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} = \mathtt{K}'\mathbf{t} \qquad \mathtt{P}' \begin{pmatrix} \mathtt{K}^{-1}\mathbf{x} \\ 0 \end{pmatrix} = \mathtt{K}'[\mathtt{R} \mid \mathbf{t}] \begin{pmatrix} \mathtt{K}^{-1}\mathbf{x} \\ 0 \end{pmatrix} = \mathtt{K}'\mathtt{R}\mathtt{K}^{-1}\mathbf{x}$$

Step 3: compute the line through the two image points using the relation $\mathbf{l}' = \mathbf{p} \times \mathbf{q}$



Compute the line through the points $\mathbf{l}' = (\mathtt{K}'\mathbf{t}) \times (\mathtt{K}'\mathtt{R}\mathtt{K}^{-1}\mathbf{x})$

Using the identity $(\mathtt{M}\mathbf{a}) \times (\mathtt{M}\mathbf{b}) = \mathtt{M}^{-\top}(\mathbf{a} \times \mathbf{b})$ where $\mathtt{M}^{-\top} = (\mathtt{M}^{-1})^{\top} = (\mathtt{M}^{\top})^{-1}$

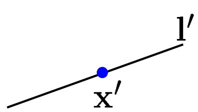
$$\mathbf{l}' = \mathtt{K}'^{-\top}\left(\mathbf{t} \times (\mathtt{R}\mathtt{K}^{-1}\mathbf{x})\right) = \underbrace{\mathtt{K}'^{-\top}[\mathbf{t}]_{\times}\mathtt{R}\mathtt{K}^{-1}}_{\mathtt{F}}\mathbf{x}$$

F is the fundamental matrix

$$\mathbf{l}' = \mathtt{F}\mathbf{x} \qquad \mathtt{F} = \mathtt{K}'^{-\top}[\mathbf{t}]_{\times}\mathtt{R}\mathtt{K}^{-1}$$

Points $\mathbf{x}$ and $\mathbf{x}'$ correspond ($\mathbf{x} \leftrightarrow \mathbf{x}'$) then $\mathbf{x}'^{\top}\mathbf{l}' = 0$

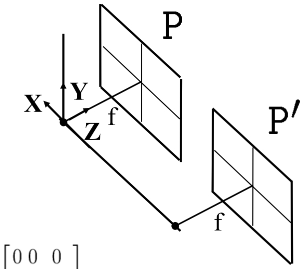
$$\mathbf{x}'^{\top}\mathtt{F}\mathbf{x} = 0$$



Example I: compute the fundamental matrix for a parallel camera stereo rig

$\mathtt{P} = \mathtt{K}[\mathtt{I} \mid \mathbf{0}] \qquad \mathtt{P}' = \mathtt{K}'[\mathtt{R} \mid \mathbf{t}]$

$$\mathtt{K} = \mathtt{K}' = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathtt{R} = \mathtt{I} \quad \mathbf{t} = \begin{pmatrix} t_x \\ 0 \\ 0 \end{pmatrix}$$



$$\mathtt{F} = \mathtt{K}'^{-\top}[\mathbf{t}]_{\times}\mathtt{R}\mathtt{K}^{-1}$$

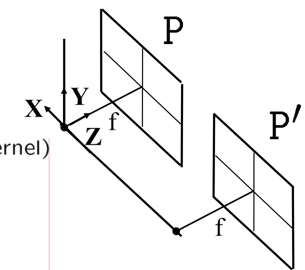
$$= \begin{bmatrix} 1/f & 0 & 0 \\ 0 & 1/f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x \\ 0 & t_x & 0 \end{bmatrix} \begin{bmatrix} 1/f & 0 & 0 \\ 0 & 1/f & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{x}'^{\top}\mathtt{F}\mathbf{x} = \begin{pmatrix} x' & y' & 1 \end{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

- reduces to $y = y'$, i.e. raster correspondence (horizontal scan-lines)

F is a rank 2 matrix

The epipole $\mathbf{e}$ is the null-space vector (kernel) of F (exercise), i.e. $\mathtt{F}\mathbf{e} = \mathbf{0}$



In this case

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

so that

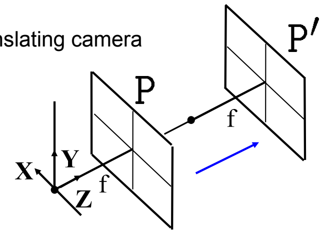
$$\mathbf{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Geometric interpretation ?

Example II: compute F for a forward translating camera

$\mathtt{P} = \mathtt{K}[\mathtt{I} \mid \mathbf{0}] \qquad \mathtt{P}' = \mathtt{K}'[\mathtt{R} \mid \mathbf{t}]$

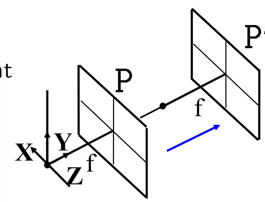
$$\mathtt{K} = \mathtt{K}' = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathtt{R} = \mathtt{I} \quad \mathbf{t} = \begin{pmatrix} 0 \\ 0 \\ t_z \end{pmatrix}$$



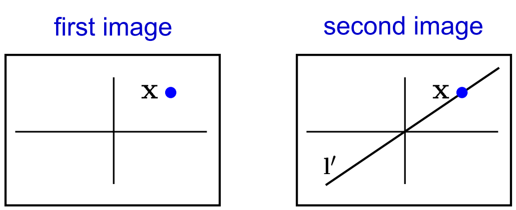
$$\begin{aligned} \mathtt{F} &= \mathtt{K}'^{-\top}[\mathbf{t}]_\times \mathtt{R}\mathtt{K}^{-1} \\ &= \begin{bmatrix} 1/f & 0 & 0 \\ 0 & 1/f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -t_z & 0 \\ t_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/f & 0 & 0 \\ 0 & 1/f & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

From $\mathbf{l}' = \mathtt{F}\mathbf{x}$ the epipolar line for the point $\mathbf{x} = (x, y, 1)^\top$ is

$$\mathbf{l}' = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$



The points $(x, y, 1)^\top$ and $(0, 0, 1)^\top$ lie on this line



Summary: Properties of the Fundamental matrix

- F is a rank 2 homogeneous matrix with 7 degrees of freedom.
- Point correspondence:
  if $\mathbf{x}$ and $\mathbf{x}'$ are corresponding image points, then $\mathbf{x}'^\top \mathtt{F}\mathbf{x} = 0$.
- Epipolar lines:
  - $\mathbf{l}' = \mathtt{F}\mathbf{x}$ is the epipolar line corresponding to $\mathbf{x}$.
  - $\mathbf{l} = \mathtt{F}^\top \mathbf{x}'$ is the epipolar line corresponding to $\mathbf{x}'$.
- Epipoles:
  - $\mathtt{F}\mathbf{e} = \mathbf{0}$.
  - $\mathtt{F}^\top \mathbf{e}' = \mathbf{0}$.
- Computation from camera matrices $\mathtt{P}, \mathtt{P}'$:
  $\mathtt{P} = \mathtt{K}[\mathtt{I} \mid \mathbf{0}],\ \mathtt{P}' = \mathtt{K}'[\mathtt{R} \mid \mathbf{t}],\ \mathtt{F} = \mathtt{K}'^{-\top}[\mathbf{t}]_\times \mathtt{R}\mathtt{K}^{-1}$

Stereo correspondence algorithms

Problem statement

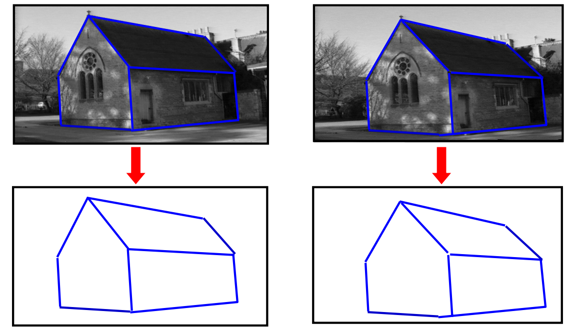
Given: two images and their associated cameras compute corresponding image points.

Algorithms may be classified into two types:

1. Dense: compute a correspondence at every pixel
2. Sparse: compute correspondences only for features

The methods may be top down or bottom up

Top down matching

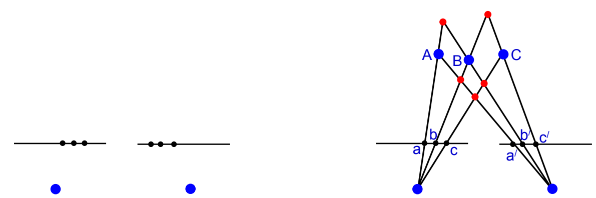


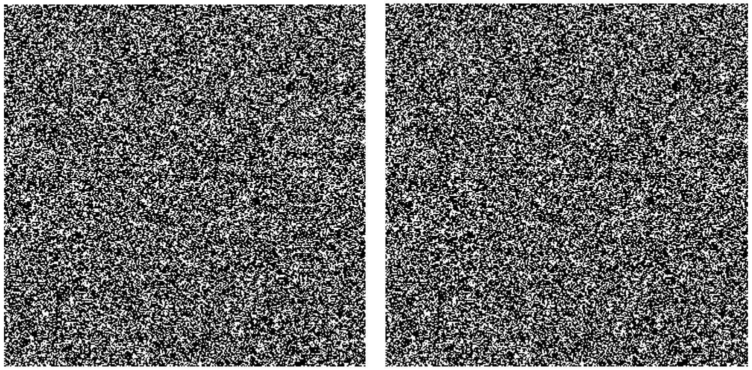
1. Group model (house, windows, etc) independently in each image
2. Match points (vertices) between images

Bottom up matching

- epipolar geometry reduces the correspondence search from 2D to a 1D search on corresponding epipolar lines

- 1D correspondence problem





cross-eye viewing random dot stereogram

Correspondence algorithms

Algorithms may be top down or bottom up – random dot stereograms are an existence proof that bottom up algorithms are possible

From here on only consider bottom up algorithms

Algorithms may be classified into two types:

1. Dense: compute a correspondence at every pixel
2. Sparse: compute correspondences only for features

Dense correspondence algorithm

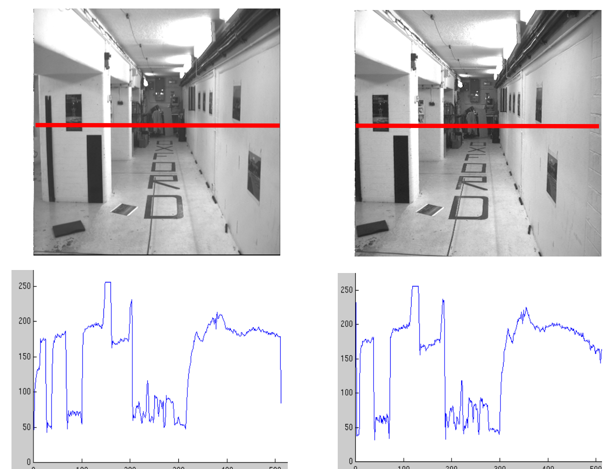
Parallel camera example – epipolar lines are corresponding rasters

Search problem (geometric constraint): for each point in the left image, the corresponding point in the right image lies on the epipolar line (1D ambiguity)

Disambiguating assumption (photometric constraint): the intensity neighbourhood of corresponding points are similar across images

Measure similarity of neighbourhood intensity by cross-correlation

Intensity profiles



- Clear correspondence between intensities, but also noise and ambiguity

Normalized Cross Correlation

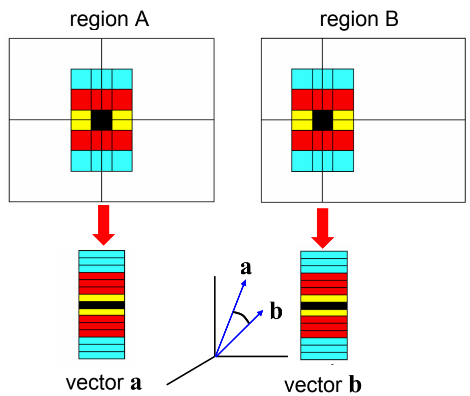
$$\text{NCC} = \frac{\sum_i \sum_j A(i,j)B(i,j)}{\sqrt{\sum_i \sum_j A(i,j)^2}\sqrt{\sum_i \sum_j B(i,j)^2}}$$

write regions as vectors

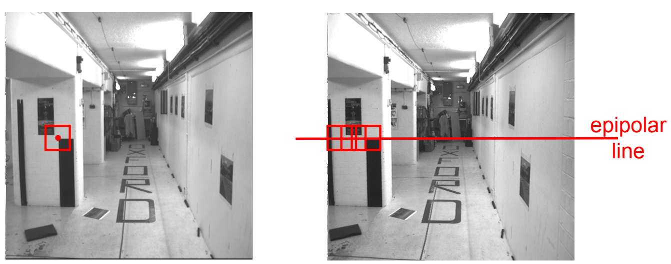
$$\mathtt{A} \rightarrow \mathbf{a}, \;\; \mathtt{B} \rightarrow \mathbf{b}$$

$$\text{NCC} = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$-1 \leq \text{NCC} \leq 1$$



Cross-correlation of neighbourhood regions



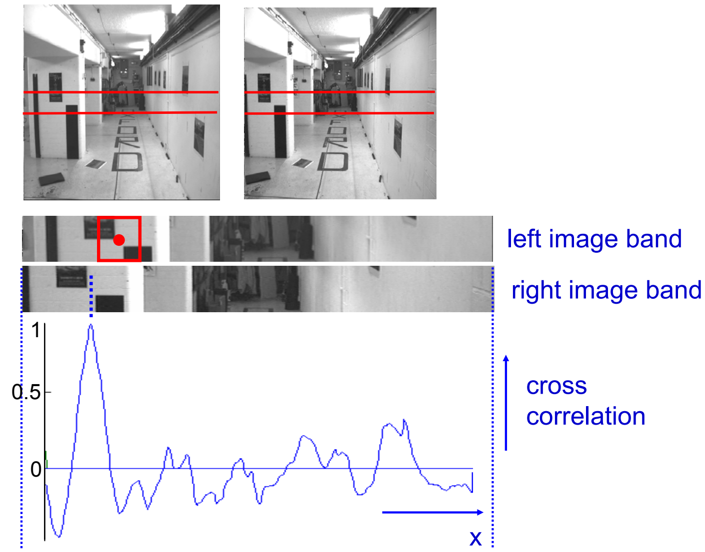
regions $\mathtt{A}, \mathtt{B}$, write as vectors $\mathbf{a}, \mathbf{b}$

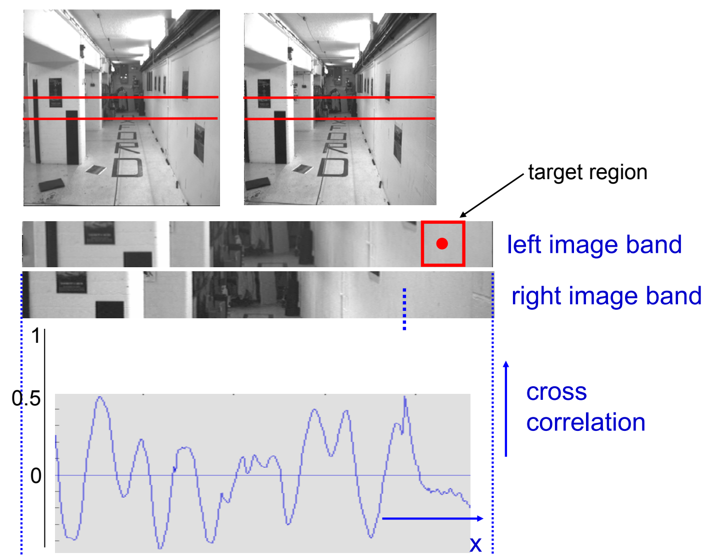
translate so that mean is zero

$$\mathtt{a} \rightarrow \mathtt{a} - \langle \mathbf{a} \rangle, \;\; \mathtt{b} \rightarrow \mathtt{b} - \langle \mathbf{b} \rangle$$

$$\text{cross correlation} = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

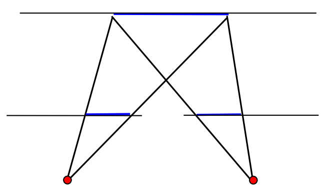
Invariant to $I \rightarrow \alpha I + \beta$
(exercise)





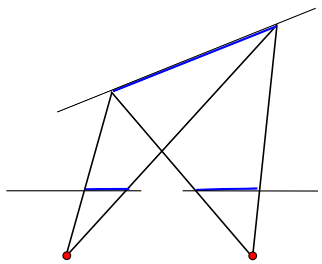
Why is cross-correlation such a poor measure in the second case?

1. The neighbourhood region does not have a "distinctive" spatial intensity distribution
2. Foreshortening effects



fronto-parallel surface

imaged length the same



slanting surface

imaged lengths differ

Sketch of a dense correspondence algorithm

For each pixel in the left image

- compute the neighbourhood cross correlation along the corresponding epipolar line in the right image
- the corresponding pixel is the one with the highest cross correlation

Parameters

- size (scale) of neighbourhood
- search disparity

Other constraints

- uniqueness
- ordering
- smoothness of disparity field

Applicability

- textured scene, largely fronto-parallel

Example dense correspondence algorithm

left image

right image

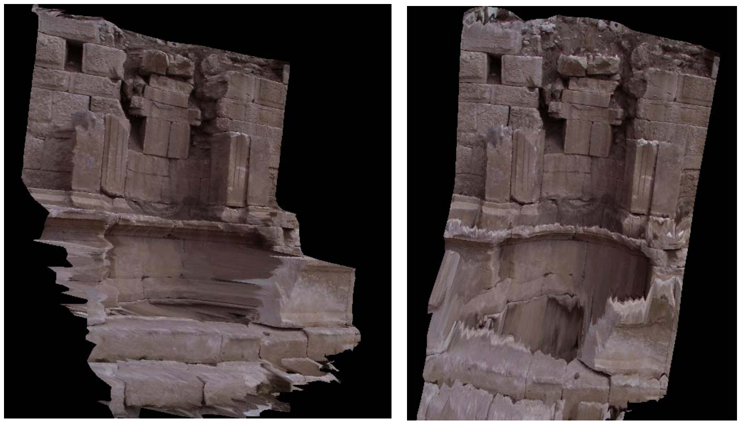
3D reconstruction

right image

depth map

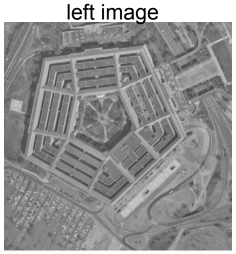
intensity = depth

Views of a texture mapped 3D triangulation

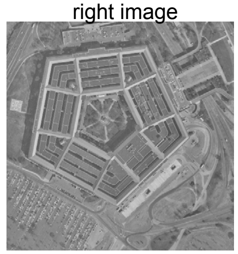


Pentagon example

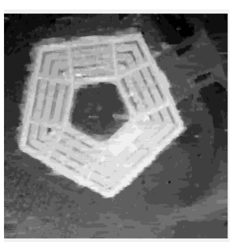
left image

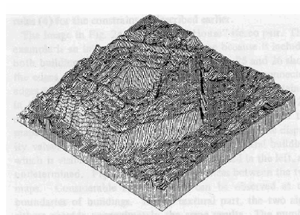


right image



range map

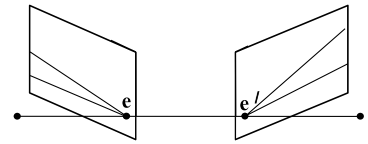


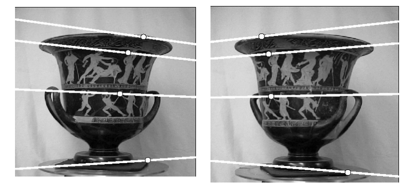


Rectification

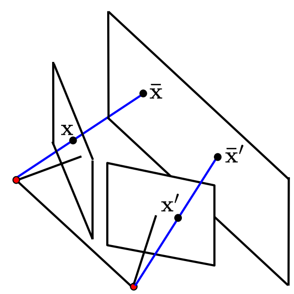
For converging cameras

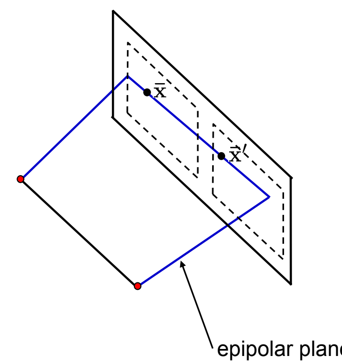
- epipolar lines are not parallel





Project images onto plane parallel to baseline





Rectification continued

Convert converging cameras to parallel camera geometry by an image mapping

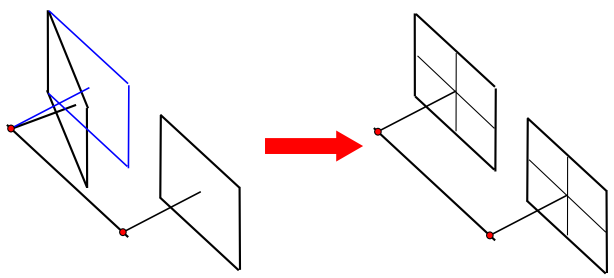


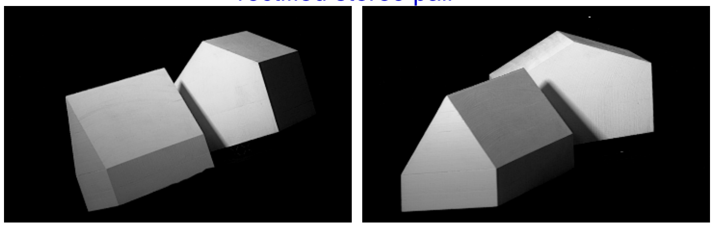
Image mapping is a 2D homography (projective transformation)

$\mathtt{H} = \mathtt{K}\mathtt{R}\mathtt{K}^{-1}$ (exercise)

Example

original stereo pair

rectified stereo pair



Example: depth and disparity for a parallel camera stereo rig

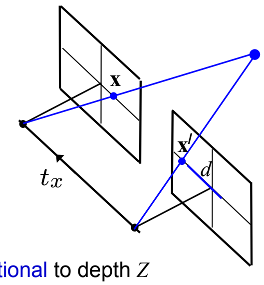
$$\mathtt{K} = \mathtt{K}' = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathtt{R} = \mathtt{I} \quad \mathbf{t} = \begin{pmatrix} t_x \\ 0 \\ 0 \end{pmatrix}$$

Then, $y' = y$, and the disparity $d = x' - x = \dfrac{f t_x}{Z}$

Derivation

$$\frac{x}{f} = \frac{X}{Z} \qquad \frac{x'}{f} = \frac{X + t_x}{Z}$$

$$\frac{x'}{f} = \frac{x}{f} + \frac{t_x}{Z}$$



Note

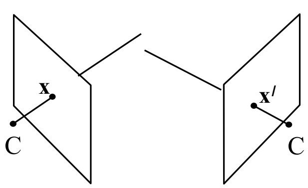
- image movement (disparity) is inversely proportional to depth Z
  as $z \rightarrow \infty$, $d \rightarrow 0$
- depth is inversely proportional to disparity

Triangulation

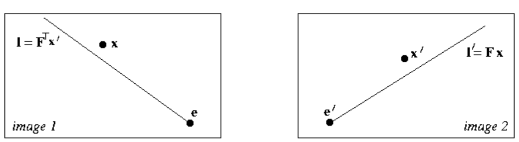
Problem statement

Given: corresponding measured (i.e. noisy) points $\mathbf{x}$ and $\mathbf{x}'$, and cameras (exact) P and P$'$, compute the 3D point $\mathbf{X}$

Problem: in the presence of noise, back projected rays do not intersect

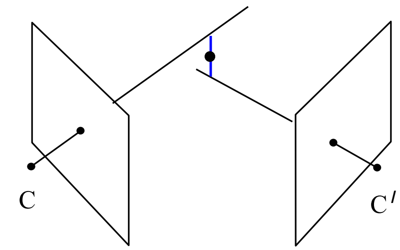


rays are skew in space



Measured points do not lie on corresponding epipolar lines

1. Vector solution



Compute the mid-point of the shortest line between the two rays

2. Linear triangulation (algebraic solution)

Use the equations $\mathbf{x} = \mathtt{P}\mathbf{X}$ and $\mathbf{x}' = \mathtt{P}'\mathbf{X}$ to solve for $\mathbf{X}$

For the first camera:

$$\mathtt{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = \begin{bmatrix} \mathbf{p}^{1\top} \\ \mathbf{p}^{2\top} \\ \mathbf{p}^{3\top} \end{bmatrix}$$

where $\mathbf{p}^{i\top}$ are the rows of $\mathtt{P}$

- eliminate unknown scale in $\lambda\mathbf{x} = \mathtt{P}\mathbf{X}$ by forming a cross product $\mathbf{x} \times (\mathtt{P}\mathbf{X}) = \mathbf{0}$

$$\begin{aligned} x(\mathbf{p}^{3\top}\mathbf{X}) - (\mathbf{p}^{1\top}\mathbf{X}) &= 0 \\ y(\mathbf{p}^{3\top}\mathbf{X}) - (\mathbf{p}^{2\top}\mathbf{X}) &= 0 \\ x(\mathbf{p}^{2\top}\mathbf{X}) - y(\mathbf{p}^{1\top}\mathbf{X}) &= 0 \end{aligned}$$

- rearrange as (first two equations only)

$$\begin{bmatrix} x\mathbf{p}^{3\top} - \mathbf{p}^{1\top} \\ y\mathbf{p}^{3\top} - \mathbf{p}^{2\top} \end{bmatrix} \mathbf{X} = \mathbf{0}$$

Similarly for the second camera:

$$\begin{bmatrix} x'\mathbf{p}'^{3\top} - \mathbf{p}'^{1\top} \\ y'\mathbf{p}'^{3\top} - \mathbf{p}'^{2\top} \end{bmatrix} \mathbf{X} = \mathbf{0}$$

Collecting together gives

$$\mathtt{A}\mathbf{X} = \mathbf{0}$$

where $\mathtt{A}$ is the 4×4 matrix

$$\mathtt{A} = \begin{bmatrix} x\mathbf{p}^{3\top} - \mathbf{p}^{1\top} \\ y\mathbf{p}^{3\top} - \mathbf{p}^{2\top} \\ x'\mathbf{p}'^{3\top} - \mathbf{p}'^{1\top} \\ y'\mathbf{p}'^{3\top} - \mathbf{p}'^{2\top} \end{bmatrix}$$

from which $\mathbf{X}$ can be solved up to scale.

Problem: does not minimize anything meaningful

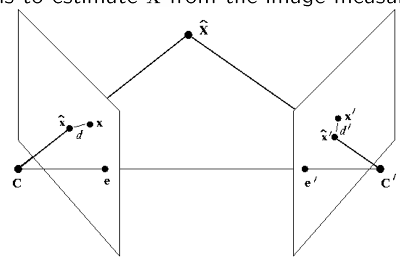
Advantage: extends to more than two views

3. Minimizing a geometric/statistical error

The idea is to estimate a 3D point $\widehat{\mathbf{X}}$ which exactly satisfies the supplied camera geometry, so it projects as

$$\hat{\mathbf{x}} = \mathtt{P}\widehat{\mathbf{X}} \qquad \hat{\mathbf{x}}' = \mathtt{P}'\widehat{\mathbf{X}}$$

and the aim is to estimate $\widehat{\mathbf{X}}$ from the image measurements $\mathbf{x}$ and $\mathbf{x}'$.



$$\min_{\widehat{\mathbf{X}}} \quad \mathcal{C}(\mathbf{x}, \mathbf{x}') = d(\mathbf{x}, \hat{\mathbf{x}})^2 + d(\mathbf{x}', \hat{\mathbf{x}}')^2$$

where $d(*, *)$ is the Euclidean distance between the points.

- It can be shown that if the measurement noise is Gaussian mean zero, $\sim N(0, \sigma^2)$, then minimizing geometric error is the Maximum Likelihood Estimate of X

- The minimization appears to be over three parameters (the position X), but the problem can be reduced to a minimization over one parameter

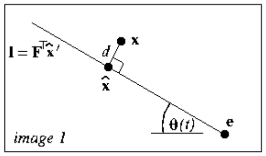
Different formulation of the problem

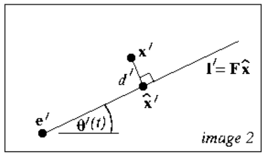
The minimization problem may be formulated differently:

- Minimize

$$d(\mathbf{x}, \mathbf{l})^2 + d(\mathbf{x}', \mathbf{l}')^2$$

- $\mathbf{l}$ and $\mathbf{l}'$ range over all choices of corresponding epipolar lines.
- $\hat{\mathbf{x}}$ is the closest point on the line $\mathbf{l}$ to $\mathbf{x}$.
- Same for $\hat{\mathbf{x}}'$.





Minimization method

- Parametrize the pencil of epipolar lines in the first image by t, such that the epipolar line is $\mathbf{l}(t)$
- Using F compute the corresponding epipolar line in the second image $\mathbf{l}'(t)$
- Express the distance function $d(\mathbf{x}, \mathbf{l})^2 + d(\mathbf{x}', \mathbf{l}')^2$ explicitly as a function of t
- Find the value of t that minimizes the distance function
- Solution is a 6th degree polynomial in t