Algebraic Properties of Noncommensurate Systems

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Abstract: Noncommensurate systems are those in which either the input vector or the output vector contains elements with different physical units. Such systems are encountered in situations where different physical quantities must be combined to achieve a given goal. An example is the control of robot manipulators where position with units of distance and orientation with units of angle must be combined. Unless proper care is taken, control systems based on current optimization techniques may yield inaccurate and misleading results. This paper examines properties of noncommensurate systems and develops several algebraic properties and requirements on the physical units involved in the control methods for such systems.

Keywords: Noncommensurate, units, robotics, control

Introduction

Control systems that combine physical quantities with different units present an additional challenge over those that concern quantities with similar physical units. Recent developments in control theory have shown that certain mathematical techniques based on eigensystems, singular value decomposition, and pseudo-inverses may lead to inconsistent and erroneous results [Schwartz 1993] [Schwartz 1995]. This paper examines the physical inconsistency of some noncommensurate systems and presents results and techniques for ensuring that mathematical developments involving noncommensurate systems are physically consistent. One type of noncommensurate system familiar to the authors is the control of robot manipulators where position with units of distance combines with orientation with units of angle and where prismatic mechanical joints and revolute joints are parts of the same robotic structure.

The theory of hybrid control of robot manipulators developed by Mason in 1978 [Mason 1978] and further expanded by Raibert in 1981 [Raibert 1981] has been shown to introduce inconsistencies [Lipkin and Duffy 1985, 1988] [Abati 1990] [Doty 1993b]. While the aim of hybrid control is to produce an optimal solution based on some optimality criterion, the end result is a solution that is non-optimal with respect to any fixed criteria and is in fact quite arbitrary. One major problem is due to the assumption and use of orthogonality in vector spaces with vectors that include elements of different physical nature. Such non-homogeneous vectors do not have a physically consistent Euclidian norm.

The robotics literature [Chiacchio 1991, Doty 1995, Klein 1987, Park 1990, Wampler 1986, Yoshikawa 1985] uses mathematical developments based on the eigenvalues, eigenvectors, or singular values of matrices that represent robotic systems with physical quantities of different nature. The results of this article indicate the invalidity, in terms of physical consistency, of such mathematical developments. In particular, it is shown that systems that combine different physical units have inconsistent eigenvalues, eigenvectors, and singular values. This research also proposes and establishes rules and guidelines for the mathematical manipulations required to maintain physical consistency in such non homogeneous system. For ease of reference, we start with a definition of noncommensurate systems.

Noncommensurate systems

We define noncommensurate systems as those involving physical quantities with different units but yet are described and controlled by physically consistent equations. Physical consistency refers here to the physical validity of all system equations in the sense of combining quantities of different units in a meaningful way. Empirically, it is about making sure that "oranges are added to oranges" and avoiding adding "apples to oranges."

In general, systems fall in one of three categories, commensurate systems, noncommensurate systems, and non-physically consistent systems. Commensurate systems are those where a single unit is involved. Commensurate systems are normally physically consistent. Noncommensurate systems, which are the topic of interest of this publication, are those that involve different physical quantities and therefore combine different units in a physically consistent way. It is important to understand that we include in this category only those systems with mixed physical units but with equations that are all physically consistent. The third category consists of systems that are not physically consistent.

As a simple example of noncommensurate system, the velocity vector V of the end-effector of a robot manipulator is related to the joint rates vector $\dot{q}$ by the manipulator Jacobian matrix J in the linear equation $V = J\dot{q}$. The velocity vector, also known as the twist, $V = [v^T, \omega^T]^T$ is composed of the linear velocity vector v, with units of distance/time, and the angular velocity vector ω, with units of angle/time. Vector $\dot{q}$, may have elements of angular velocity (for revolute joints), and elements of linear velocity for prismatic joints. Physical consistency is achieved by proper choice of units for the elements of the system matrix J. To illustrate this point, consider the following example.

Example 1. The robot manipulator defined by the Denavit-Hartenberg parameters [Denavit 1955] in Table 1 and described in Figure 1 is redundant if only position is considered in the task space.

Table 1: DH-parameters

Joint	d	a	α	θ
1	0	a_1	0	θ_1
2	d_2	0	90°	0
3	0	a_3	90°	θ_3
4	0	a_4	0	θ_4

Forward kinematics computations yield the end-effector position vector

$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} a_4(c_1c_3c_4 + s_1s_4) + a_3c_1c_3 + a_1c_1 \\ a_4(s_1c_3c_4 - c_1s_4) + a_3s_1c_3 + a_1s_1 \\ a_4s_3c_4 + a_3s_3 + d_2 \end{bmatrix},$$

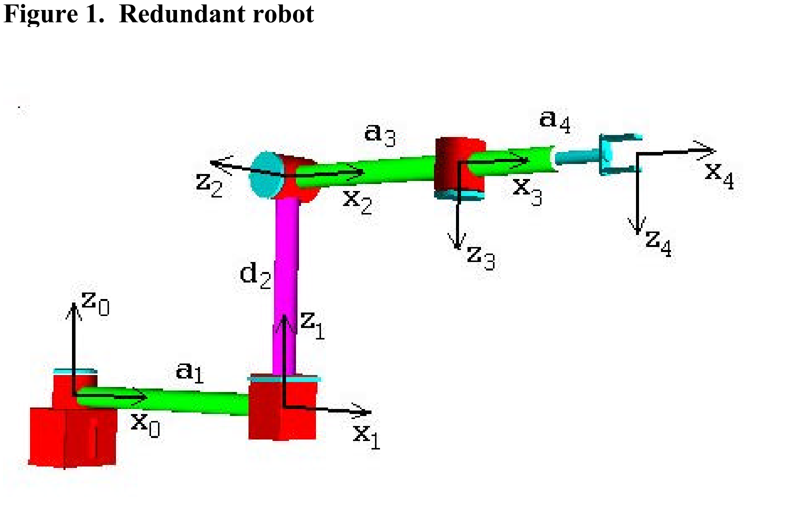
where we used the customary notation $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$. Since position is determined by a minimum of three degrees of freedom, the robot under consideration, with its four joints, is redundant with respect to position. This robot is chosen because it contains prismatic and rotational joints. The linear velocity vector of the end-effector frame is

$$v = J\dot{q}, \quad \text{where,} \quad v = \left[\frac{dp_x}{dt}, \frac{dp_y}{dt}, \frac{dp_z}{dt}\right]^T \text{ and}$$

$\dot{q} = [\dot{\theta}_1 \ \dot{d}_2 \ \dot{\theta}_3 \ \dot{\theta}_4]^T$ is the joint velocity vector, and J is the Jacobian matrix of the robot. The Jacobian matrix is expressed as:

$$J = \begin{bmatrix} -s_1(c_3a_4c_4 + a_3c_3) + c_1a_4s_4 - a_1s_1 & 0 & -c_1(s_3a_4c_4 + a_3s_3) & -c_1c_3a_4s_4 + s_1a_4c_4 \\ c_1(c_3a_4c_4 + a_3c_3) + s_1a_4s_4 + a_1c_1 & 0 & -s_1(s_3a_4c_4 + a_3s_3) & -s_1c_3a_4s_4 - c_1a_4c_4 \\ 0 & 1 & c_3a_4c_4 + a_3c_3 & -s_3a_4s_4 \end{bmatrix}.$$

Figure 1. Redundant robot



In terms of units, all elements of matrix J have units of length except for the second column whose elements are unitless numbers. This makes $V = J\dot{q}$ physically consistent since all elements of vector $\dot{q}$ are in units of inverse time (time^{-1}) except for the second element, which has units of velocity (length/time).

Manipulability studies rely on the product JJ^T whose elements are listed here:

$$
\begin{aligned}
JJ^T_{11} &= (-s_1(c_3 a_4 c_4 + a_3 c_3) + c_1 a_4 s_4 - a_1 s_1)^2 + c_1^2(-s_3 a_4 c_4 - a_3 s_3)^2 + (-c_1 c_3 s_4 + s_1 c_4)^2 a_4^2 \\
JJ^T_{12} &= (-s_1(c_3a_4c_4 + a_3c_3) + c_1a_4s_4 - a_1s_1)\,(c_1(c_3a_4c_4 + a_3c_3) + s_1a_4s_4 + a_1c_1) + c_1(-s_3a_4c_4 - a_3s_3)^2 s_1 \\
&\quad + (-c_1c_3s_4 + s_1c_4)\,(-s_1c_3s_4 - c_1c_4)a_4^2 \\
JJ^T_{12} &= c_1(-s_3a_4c_4 - a_3s_3)\,(c_3a_4c_4 + a_3c_3) - (-c_1c_3s_4 + s_1c_4)\,s_3s_4a_4^2 \\
JJ^T_{21} &= (-s_1(c_3a_4c_4 + a_3c_3) + c_1a_4s_4 - a_1s_1)\,(c_1(c_3a_4c_4 + a_3c_3) + s_1a_4s_4 + a_1c_1) + c_1(-s_3a_4c_4 - a_3s_3)^2 s_1 \\
&\quad + (-c_1c_3s_4 + s_1c_4)(-s_1c_3s_4 - c_1c_4)a_4^2 \\
JJ^T_{22} &= (c_1(c_3a_4c_4 + a_3c_3) + s_1a_4s_4 + a_1c_1)^2 + s_1^2(-s_3a_4c_4 - a_3s_3)^2 + (-s_1c_3s_4 - c_1c_4)^2 a_4^2 \\
JJ^T_{23} &= s_1(-s_3a_4c_4 - a_3s_3)\,(c_3a_4c_4 + a_3c_3) - (-s_1c_3s_4 - c_1c_4)s_3s_4a_4^2 \\
JJ^T_{31} &= c_1(-s_3a_4c_4 - a_3s_3)\,(c_3a_4c_4 + a_3c_3) - (-c_1c_3s_4 + s_1c_4)s_3s_4a_4^2 \\
JJ^T_{32} &= s_1(-s_3a_4c_4 - a_3s_3)(c_3a_4c_4 + a_3c_3) - (-s_1c_3s_4 - c_1c_4)s_3s_4a_4^2 \\
JJ^T_{33} &= 1 + (c_3a_4c_4 + a_3c_3)^2 + s_3^2\, a_4^2\, s_4^2
\end{aligned}
$$

Careful examination of the units involved in the above equations indicates that each element of JJ^T is actually the sum of a quantity of length-squared added to a unitless quantity. The unitless quantity is 0 in all elements of JJ^T except in JJ^T_{33} where it is equal to 1. This example shows that the matrix JJ^T does not make physical sense.

The determinant of JJ^T (or its square root) is typically used as a measure of manipulability at the current robot configuration. Choosing rotational joint values of $q_1 = \frac{p}{6}, q_3 = \frac{p}{6},\ and\ q_4 = -\frac{p}{6}$ rather arbitrarily and computing the determinant, d, of the matrix JJ^T defined above yields:

$$
\begin{aligned}
d = {} & 1.125\ a_4a_1a_3^2 + .6495\ a_4a_3^3 + .75\ a_4^3a_3^2a_1 + .3248\ a_4^3a_3^3 + .7307\ a_4^3a_3 \\
& + .25\ a_1^2a_3^2a_4^2 + .433a_3^3a_1 + .1406\ a_4^4a_3^2 + .1875\ a_4^2a_3^4 + 1.172\ a_4^2a_3^2 \\
& + .375\ a_4^2a_1^2 + .1875\ a_3^4 + .25\ a_1^2a_3^2 + .2578\ a_4^4 + .433\ a_3^3a_1a_4^2 \\
& + .4330\ a_1^2a_4^3a_3 + .3248\ a_4^4\ a_3a_1 + .1875\ a_1^2a_4^4 + .1875\ a_4^3a_1 + 1.299\ a_4^2a_1a_3 \\
& + .4330\ a_1^2a_4a_3
\end{aligned}
$$

Again examination of the units involved shows that d is not physically consistent since terms of units length-to-the-sixth-power are added to terms with units of length-to-the-fourth-power!

Manipulability theory applied to redundant robots does not lead to consistent results as shown in this example. The goals of this article is to examine the problems associated with systems involving physical quantities of different units and establish a few ground rules for such systems.

Linear noncommensurate systems

A vector of elements of unlike physical units is defined as a noncommensurate vector, also known as a compound or non-homogenous vector. A linear system is described by a linear equation of the form

$$y = Ax\,, \tag{1}$$

where $x=[x_1,x_2,...,x_n]^T$ and $y=[y_1,y_2,...,y_m]^T$ are vectors and $A=\{a_{ij}\}_{i=1,...,n;\ j=1,...,m}$ is an n×m matrix. Let us assume that x and y in Eq. (1) are noncommensurate vectors. In this section we will determine conditions on the coefficient matrix A for the system (1) to be physically consistent. The i^{th} component of vector y is given by:

$$y_i=\sum_{j=1}^{m} a_{ij}x_j \qquad (2)$$

For consistency, we require that

$$units(y_i)=units(a_{ij})\ \ units(x_j) \qquad (3)$$

for all i,j.

Using two distinct terms of the sum (2), for two elements of y, we get

$$z_i=a_{ij}x_j+a_{ik}x_k \qquad (4)$$

$$z_l=a_{lj}x_j+a_{lk}x_k \qquad (5)$$

for all *i, j, k, l,* where $units(z_i)=units(y_i)$. Manipulating equations (4) and (5) yields

$$a_{ik}z_l-a_{lk}z_i=(a_{ik}a_{lj}-a_{ij}a_{lk})\ x_j\ . \qquad (6)$$

Physical consistency demands that

$$units(a_{ik})\ \ units(a_{lj})=units(a_{ij})\ \ units(a_{lk}) \qquad (7)$$

or

$$units\left[\frac{a_{ik}}{a_{ij}}\right]=units\left[\frac{a_{lk}}{a_{lj}}\right],\ \text{for}\ a_{ij}\neq 0\ \text{and}\ a_{lj}\neq 0\ . \qquad (8)$$

In other words, if m-2 columns and n-2 rows are eliminated, the determinant of the remaining 2 x 2 matrix must be physically consistent for the system to be noncommensurate. This result is summarized in a theorem as a verification test for the commensurate or noncommensurate nature of a matrix A

Theorem 1. The linear equation $y=Ax$, where x and/or y are noncommensurate vectors, describes a noncommensurate system if and only if the determinant of any 2×2 matrix obtained by eliminating all but 2 rows and all but two columns of A is physically consistent.

Using three terms from the sum (2) for three elements of vector y, we get three equations of the form of (4) and (5). Manipulating these three equations leads to the result that physical consistency requires that the determinant of any 3×3 matrix obtained by the elimination of m-3 rows and n-3 columns must be physically consistent for the system to be noncommensurate.

By induction, we provide that a condition for the system of equation (1) to be noncommensurate is that the determinant of any i×i matrix obtained by eliminating (m-i) rows and (n-i) columns must be physically consistent.

Another requirement may be obtained by partially expanding Eq. (1) as follows:

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}=\sum_{j=1}^{m}\begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}x_j. \qquad (9)$$

It follows that the units of any two columns of A must be proportional for consistency. This is another expression of the result of Eq. (8) and a simple way to visually evaluate the possible noncommensurate nature of matrix *A*.

Corollary 1.1: A matrix *A*, with elements of different units, is noncommensurate if and only if the units of any two columns of the matrix are proportional.

Note that in the case of commensurate matrices, all elements have the same unit, therefore corollary 1.1 essentially indicates that a matrix A whose columns have proportional units will lead to a physically consistent system described by Eq. (1).

Eigenvalues in Noncommensurate Systems

The treatment of equation (1) in many applied fields requires computation and use of eigenvalues and eigenvectors. In this section, the physical consistency of eigenvalues and eigenvectors is examined and conditions on the coefficient matrix *A* are established for physical consistency. Let *e* be an eigenvector of matrix *A*, and λ the corresponding eigenvalue, then

$$Ae=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}=\lambda\ e\ . \qquad (10)$$

Expanding:

$$\begin{aligned} a_{11}e_1 + a_{12}e_2 + ... + a_{1n}e_n &= \lambda e_1 \\ a_{21}e_1 + a_{22}e_2 + ... + a_{2n}e_n &= \lambda e_2 \\ &\vdots \\ a_{n1}e_1 + a_{n2}e_2 + ... + a_{nn}e_n &= \lambda e_n \end{aligned} \quad (11)$$

Since meaningful addition must be with terms of identical units, the following theorem on the units involved must hold:

Theorem 2: The equation $Ax = \lambda x$ is physically consistent if and only if

$$units[a_{kj}]\, units[x_j] = units[\lambda]\, units[x_k]$$

for all j and k.

Direct consequences of this theorem are:

Corollary 2.1: The equation $Ax = \lambda x$ is physically consistent only if $units[\lambda] = units[a_{ii}]$ for all i.

Corollary 2.2: The equation $Ax = \lambda x$ is physically consistent if and only if

$$units[a_{kj}]units[a_{jk}] = units[a_{ii}^2]$$

for all i, j, and k.

From corollary 2.1, any matrix A with a physically consistent eigenvalue equation must have diagonal elements with the same physical units and all its eigenvalues must have those same units.

Singular Value Decomposition

Many processes in system control rely on a singular value decomposition. In this section, the physical consistency of singular values in noncommensurate systems is examined. The singular values of matrix A are the non-negative square roots of the non-zero eigenvalues of the square matrices $A A^T$ and $A^T A$. A test similar to those discussed above on these matrix products will determine if the SVD of matrix A is physically consistent.

Let $B = AA^T$ or $B = A^T A$ and x and λ be an eigenvector and eigenvalue of B respectively. From theorem 1 above, physical consistency requires that

$$units[b_{kj}]\, units[x_j] = units[\lambda]\, units[x_k], \quad (12)$$

where $B = \{b_{ij}\}_{i=\text{row index},\ j=\text{column index}}$.

For diagonal elements of B, Eq. (12) yields $units[b_{kk}] = units[\lambda]$ which indicates that all diagonal elements of B must have the same units. If A is an n×m matrix, then the diagonal elements of B are given by

$$b_{kk} = \begin{cases} \sum_{j=1}^{m} a_{kj}^2 \text{, for } B = AA^T \\ \sum_{j=1}^{n} a_{jk}^2 \text{, for } B = A^T A \end{cases}, \text{ for all k.} \quad (13)$$

Therefore, all the elements in the k[th] row (for $B = AA^T$) or the k[th] column (for $B = A^T A$) must have identical units. Since all diagonal elements must also have the same units, the Singular Value Decomposition is physically consistent only for commensurate systems.

Theorem 3: Noncommensurate systems do not have a physically consistent singular value decomposition.

Theorem 3 indicates that any development based on the singular value decomposition of a matrix with different units is invalid in the sense of physical consistency.

Example 2: The field of robotics offers an example of the invalid use of the eigensystem and SVD. Consider a redundant robot manipulator described by the DH-parameters of Table 2. This robot has three positional prismatic joints and a 4-joint wrist.

Table 2: DH-parameters for 3P-4R robot

Joint	Type	d	a	α	θ
1	P	d_1	0	0	π/2
2	P	d_2	0	π/2	π/2
3	P	d_3	0	0	0
4	R	0	0	θ_4	π/2
5	R	0	0	θ_5	π/2
6	R	0	0	θ_6	π/2
7	R	0	0	θ_7	0

The Jacobian matrix J for this robot is not square and the pseudo-inverse is typically used in solving kinematic equations. The matrix JJ^T is used in the computation of the pseudo-inverse of J [Ben-Israel 1974] and a manipulability measure is defined by

$$\rho = \sqrt{\det(JJ^T)}\,. \quad (14)$$

Further, Yoshikawa [Yoshikawa 1985, Yoshikawa 1990] and others [Klein 1987, Park 1990] define a manipulability ellipsoid with principal axes in the directions of the eigenvectors of JJ^T. Each axis is given a length of $\sqrt{1/\lambda_i}$ where λ_i is an eigenvalue of JJ^T.

For an all revolute manipulator, the matrix JJ^T is a 6×6 matrix that is physically consistent and can be viewed as formed with 3×3 block matrices B_{11}, B_{12}, B_{21}, and B_{22}, i.e.,

$$JJ^T = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \quad (15)$$

Each block matrix has its own physical units. For an all-revolute robot arm, B_{11} has elements with units of length-squared (length^2), B_{12} and B_{21} have units of length, and B_{22} is unitless. Let L stand for units of length and U for Unitless. Then the units of the main diagonal of JJ^T are represented by [L^2, L^2, L^2, U,U,U]. Since the units of the elements of the main diagonal are not all the same, corollary 1 of theorem 1 above states that JJ^T does not support a physically consistent eigensystem. This invalidates any development based on the eigensystem of the matrix JJ^T!

If the manipulator is not all revolute, then the JJ^T matrix is itself physically inconsistent since B_{11}, defined in Eq. (15) has units of [L^2+U]. Example 1 discussed above supports this discussion.

Conclusion

This paper has addressed the problem of noncommensurate systems from the point of view of physical consistency. An analysis of the units involved in the mathematical derivations routinely performed in the control of noncommensurate systems provides a few tests for physical consistency that can be readily applied to linear hybrid systems where physical quantities of different nature are combined in the controls equations. Particularly, the use of eigenvalues, eigenvectors, and singular values in the control of noncommensurate systems is shown to lead to physically inconsistent results. This paper uses examples in robotics to illustrate the concepts discussed and presented, but the results apply to any control algorithm based on equations derived through the use of eigenvalues, eigenvectors, or singular values of matrices whose elements have different units.

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