ALGEBRAIC PROPERTIES OF NONCOMMENSURATE SYSTEMS AND THEIR APPLICATIONS IN ROBOTICS

By

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To My Wife Gabriella
&
To My Parents Marilyn and Seymour

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KEY TO SYMBOLS

Symbol or Variable	Definition
$\dagger$	for $X^\dagger$, the Moore-Penrose pseudo-inverse of X
$\#$	for $X^\#$, the weighted generalized-inverse of X
$\lvert\cdot\rvert$	for $\lvert x\rvert$ where x is a vector, $\lvert x\rvert = \sqrt{x \odot x}$
$\lvert\cdot\rvert_M$	for $\lvert x\rvert_{M_x}$ where x is a vector, $\lvert x\rvert = \sqrt{x \odot M_x x}$
$\times$	for $x \times y$, the vector cross product of vectors x and y
$\odot$	for $x \odot y$, the inner (or dot) product of vectors x and y
$\circ$	for $X \circ Y$, the klein (or reciprocal) product of screws X and Y
$\oplus$	for $\mathcal{X} \oplus \mathcal{Y}$, the direct sum of the subspaces $\mathcal{X}$ and $\mathcal{Y}$
$\overset{M}{\oplus}$	for $\mathcal{X} \overset{M}{\oplus} \mathcal{Y}$, the direct sum of the M-orthogonal subspaces $\mathcal{X}$ and $\mathcal{Y}$
$\overset{?}{=}$	possibly equal, often physically inconsistent
$\overset{def}{=}$	defined as
$\overset{N}{=}$	numerically equal to
$(\cdot)_{(i,j)}$	for matrix $X_{(i,j)}$, element of X in i-th row, j-th column
$(\cdot)_{(\cdot,j)}$	for matrix $X_{(\cdot,j)}$, the j-th column of X
$(\cdot)_{(i,\cdot)}$	for matrix $X_{(i,\cdot)}$, the i-th row of X
$[\cdot]_{r,c}$	for $[X]_{r,c}$,an $r \times c$ matrix with all units identical to the units of L
$[0]_{r,c}$	$r \times c$ matrix of zeros
$[\cdot]_b$	for $[X]_b$, matrix where the column vectors constitute a basis for $\mathcal{X}$
$[\cdot]^\tau$	the transpose operator
0_n	zero vector of dimension n
α	angle between successive joint axes projected on plane with common normal used in D-H parameterization
Δ	orthogonal 6×6 matrix that converts between ray and axis coordinates
θ	angle about a joint axes used in D-H parameterization
κ_i	$\cos(\alpha_i)$
σ_i	$\sin(\alpha_i)$
τ	the generalized-force vector containing n joint forces and/or joint torques corresponding to prismatic and/or revolute joints
ω	angular velocity 3-vector
A	wrench coordinate transformation matrix
a	perpendicular distance between successive joint axes used in D-H parameterization

Symbol or Variable	Definition
B	skew-symmetric 3×3 translation matrix of b
b	translation 3-vector
c_{i+j}	$\cos(\theta_i + \theta_j)$
$\mathcal{D}$	defect manifold
d	distance along joint axis used in D-H paramtetrization
E_x	matrix such that $\mathcal{X} E_x$ is the column-reduced echelon form of $\mathcal{X}$
f	force 3-vecor
G	twist coordinate transformation matrix
I_b	body's inertia tensor at the center-of-mass expressed in principal corrdinates—a diagonal matrix
I_j	$j \times j$ identity matrix
J	manipulator Jacobian that transforms joint rates into twists, $V = J\dot{q}$
J_ω	first three rows of J, such that $v = J_v \dot{q}$
J_v	rows four through six of J, such that $\omega = J_\omega \dot{q}$
$[L]_{r,c}$	$r \times c$ units matrix with all units of length
n	number of joints in manipulator
n	moment of force 3-vector
$\mathrm{Null}[A]$	null space of matrix A, *i.e.,* all x such that $Ax = 0$
$\mathcal{Q}$	joint-rates vector space
R	3x3 rotation matrix
$\mathcal{R}$	radical subspace
$\Re^m$	commensurate m-space over reals
$\mathrm{Range}[A]$	range space of matrix A, *i.e.,* all y such that $y = Ax$
S or S_i	rotation vector of screw i
S_0 or S_{0i}	translation vector of screw i
S_q	change of units scaling matrix for joint rates
S_v	change of units scaling matrix for twists
s_{i+j}	$\sin(\theta_i + \theta_j)$
$\mathcal{T}$	generalized (joint) forces vector space
$[U]_{r,c}$	$r \times c$ unitless units matrix
$\mathrm{units}[\cdot]$	the physical dimensions of the matrix inside the brackets
V	twists in Plücker ray coordinates, $V = [v^\tau,\ \omega^\tau]^\tau$
$\mathcal{V}$	twists screw space
$\mathcal{V}_f$	twists of freedom subspace
$\mathcal{V}_{nf}$	twists of nonfreedom manifold
v	linear velocity 3-vector
W	wrench in Plücker axis coordinates, $W = [f^\tau,\ n^\tau]^\tau$
$\mathcal{W}$	wrenches screw space
$\mathcal{W}_c$	wrenches of constraint subspace
$\mathcal{W}_{nc}$	wrenches of nonconstraint manifold
$\hat{z}$	unit vector in z direction ($[0,\ 0,\ 1]^\tau$)

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Several algebraic properties are given for systems in which either or both the input and output vectors have elements with different physical units. The conditions on linear transformation A for a physically consistent noncommensurate system, $u = Ax$, are given. Linear noncommensurate systems do not generally have eigenvalues and eigenvectors. The requirements for a noncommensurate system to possess a physically consistent eigensystem are presented. It is also shown that noncommensurate linear systems do not have a physically consistent singular value decomposition.

The manipulator Jacobian maps possibly noncommensurate robot joint-rate vectors into noncommensurate twist vectors. The inverse velocity problem is often solved through the use of the pseudo-inverse of the Jacobian. This solution is generally scale and frame dependent. The pseudo-inverse solution is physically inconsistent, in general, requiring the addition of elements of unlike physical units. For some manipulators there may exist points—called decouple points—at which the pseudo-inverse of the Jacobian is physically consistent for all frames at these points.

In decouple frames, the pseudo-inverse is shown to be equivalent to the weighted generalized-inverse with identity metrics. An entire class of nonidentity metrics used with the weighted generalized-inverse are shown to give identical solutions to the pseudo-inverse solution at decouple points.

At decouple points, the twist and wrench spaces can be decomposed into two metric-independent subspaces. This decomposition is accomplished with kinestatic filtering projection matrices.

The Mason/Raibert hybrid control theory of robotics is shown to be useful only for frames located at decouple points and is not optimal in any objective sense.

The current manipulability theory, which depends on the eigensystem of various functions of the Jacobian, is shown to be invalid.

Two new classes of manipulators are introduced, self-reciprocal manipulators and decoupled manipulators. The twists of freedom of a self-reciprocal manipulator are reciprocal. The class of self-reciprocal manipulators consists of planar manipulators, spherical manipulators, and prismatic-jointed manipulators. Decoupled manipulators are shown to decouple at every point. The manipulators of this class are planar manipulators, prismatic-jointed manipulators, and SCARA-type manipulators. Results that are generalized from decoupled manipulators often prove to be invalid for manipulators that do not decouple at every point.

CHAPTER 1
INTRODUCTION

Optimum, according to Webster [58], means "best; most favorable." In real physical systems, to say a solution is optimum or optimal one must specify the criteria for optimality.

The theory of hybrid control of manipulators developed by Mason in 1978 [41, 40] and then tested and expanded by Raibert in 1981 [51] has been shown by Lipkin and Duffy [37, 36] and others [1, 19] to be erroneous. Lipkin and Duffy explain that the failure of Mason and Raibert's hybrid control theory (MRHCT) is in their use of orthogonality. In MRHCT, the orthogonality of two vectors with terms of unlike units is used when it is easily seen that the inner product of these vectors in not invariant to scaling. Because so many authors continued to use MRHCT, Duffy [22] found it necessary to write an editorial debunking this theory.

The problem with MRHCT, in this author's view, is that the terms of their optimal solution were not sufficiently defined. An exploration of the meaning of their optimal solution would have shown that the solution is based on minimizing the Euclidean norms of two non-Euclidean vectors.

In 1989, Doty noticed and eventually published research [14, 19] that the Moore-Penrose pseudo-inverse solution in the robotics inverse velocity problem gives results that are dependent on the frames of reference. Doty's algebraic viewpoint, together with Duffy and Lipkin's geometric results using screw theory, suggested a further investigation of the possible non-invariance of solution techniques in several areas of robotics and applied mathematics in general.

This dissertation is based in part on correcting the inappropriate use of the pseudo-inverse in the field of robotics. Researchers such as Doty [18], Duffy [22], Lipkin and Duffy [37, 36], Lipkin [35], Griffis [26], and Schwartz [54, 53] have shown the fallacy of incorrectly applying optimization techniques to robotics problems without a judicious investigation of the underlying metrics incorporated. This dissertation intends to formalize and explain these problems and offer consistent solutions and interpretations of these solutions.

Each of these problems involves solving a set of linear equations which by some manipulation can be put in the form $u = Ax$, where A is nonsquare or singular. More often than not, a multitude of robotics researchers including [12, 23, 29, 32, 38, 39, 43, 44, 45] have solved these problems by using the pseudo-inverse. The inconsistent results generated through the use of the pseudo-inverse (without a metric or metrics) are explained in this dissertation.

The robotics literature [10, 31, 46, 57, 59, 60] also makes use of the eigenvalues, eigenvectors, or singular values of matrices whose eigenvalues and singular values are not invariant to changes in scale or coordinate transformations, and are therefore not true "eigensolutions". The eigensolution problem is also discussed in this dissertation.

The basic mathematics and terminology of robotics and screw theory necessary for an understanding of the issues discussed will be introduced in this chapter. There is no original work in this chapter other than some basic definitions with regard to noncommensurate systems. Since a general understanding of the Euclidean vector norm, the pseudo-inverse, the weighted generalized-inverse, eigenvalues, eigenvectors, and singular value decomposition are paramount to understanding this dissertation, these topics will also be presented and examples (with references) of their use in robotic systems will be given in this chapter.

1.1 Noncommensurate Vector Spaces

Systems involving elements of different physical units are defined here as *noncommensurate systems*. Robotics systems are noncommensurate when they deal with both position and orientation or have both revolute and prismatic joints. A vector of elements of unlike physical units is defined as a *noncommensurate vector*. (The noncommensurate vector is also called a compound vector [14, 53] and non-homogeneous vector [15].)

In robotics, the equation that relates joint velocities to twists (1.1) describes a noncommensurate system,

$$V = J\dot{q} \quad . \tag{1.1}$$

The manipulator joint-rate vector is

$$\dot{q} = [\dot{q}_1,\ \dot{q}_2,\ \ldots,\ \dot{q}_n]^\tau \quad , \tag{1.2}$$

where n represents the total number of revolute and prismatic joints of the manipulator. The manipulator's instantaneous *twist* vector,

$$V = [v^\tau,\ \omega^\tau]^\tau \quad , \tag{1.3}$$

is composed of the linear velocity $v = [v_x,\ v_y,\ v_z]^\tau$ and the angular velocity $\omega = [\omega_x,\ \omega_y,\ \omega_z]^\tau$. The Jacobian J is a $6 \times n$ matrix, where 6 is the number of coordinates necessary to describe the position and orientation of a body in space.

The twist represents a noncommensurate vector since the units of v and ω differ. When the manipulator has both revolute and prismatic joints, the joint-rate vector is also noncommensurate and the manipulator is called a *noncommensurate manipulator*.

The vector ${}^{i,m}V_{p,k}$ represents the twist of a point p, fixed to frame k, and expressed in frame i coordinates with respect to a fixed frame m. Since the Jacobian ${}^{i,m}J_{p,k}$ has columns that are also twists, the superscript i and m and the subscripts p and k

have the same interpretations as in ${}^{i,m}V_{p,k}$. When the subscripts p and k and the superscript m are omitted in ${}^{i}V$ and ${}^{i}J$, it is understood that k is the end-effector frame n of an n-jointed manipulator, m is the base frame (frame 0), and point p is at the origin of frame i, the frame of expression (${}^{i}V = {}^{i,0}V_{i,n}$).

To transform twists or Jacobians to representations in different frames, the twist coordinate transformation matrix G is used,

$$ {}^{i}G_{j}^{p,q} = \begin{bmatrix} {}^{i}R_{j} & {}^{i}B_{p,q}\,{}^{i}R_{j} \\ [0]_{3,3} & {}^{i}R_{j} \end{bmatrix} \quad , \tag{1.4} $$

where $[0]_{3,3}$ is a 3×3 matrix of zeros and ${}^{i}R_{j}$ is a rotation transformation which rotates a vector from frame j into frame i. Since rotation matrices are orthogonal, the inverse is equal to the transpose, *i.e.,*

$$ {}^{i}R_{j}^{-1} = {}^{i}R_{j}^{\tau} = {}^{j}R_{i} \quad . \tag{1.5} $$

(By convention, the term "orthogonal matrix" refers to matrices with orthonormal columns [56].) The matrix ${}^{i}B_{p,q} = [{}^{i}b_{p,q}\times]$ is a skew symmetric matrix that represents translation from point p to q expressed in frame i. The B matrix is the matrix-form of the vector cross-product, *i.e.,* $Bc = b \times c$, where b and c are arbitrary 3-vectors and B is defined as

$$ {}^{i}B_{p,q} = \begin{bmatrix} 0 & -{}^{i}b_{z} & {}^{i}b_{y} \\ {}^{i}b_{z} & 0 & -{}^{i}b_{x} \\ -{}^{i}b_{y} & {}^{i}b_{x} & 0 \end{bmatrix} \quad . \tag{1.6} $$

The vector ${}^{i}b_{p,q} = [{}^{i}b_{x},\ {}^{i}b_{y},\ {}^{i}b_{z}]^{\tau}$ is a position vector from point p to point q expressed in frame i coordinates.

Since B is skew symmetric, it has the following properties:

$$ B_{q,p} = -B_{p,q} \quad , \tag{1.7} $$

$$ (B_{p,q})^{\tau} = -B_{p,q} \quad \text{, and} \tag{1.8} $$

$$ {}^{i}R_{j}\,{}^{j}B_{p,q} = {}^{i}B_{p,q}\,{}^{i}R_{j} \quad . \tag{1.9} $$

With the above equations it is easily shown that

$$\left({}^{i}G_{j}^{p,q}\right)^{-1} = {}^{j}G_{i}^{q,p} \quad . \tag{1.10}$$

Note that $\left({}^{i}G_{j}^{p,q}\right)^{\tau} \neq {}^{j}G_{i}^{q,p}$.

The expressions for the frame transformations of twists and Jacobians are

$$\begin{aligned} {}^{i}V_{p,k} &= \left({}^{i}G_{j}^{p,q}\right) {}^{j}V_{q,k} \quad \text{, and} \qquad (1.11)\\ {}^{i}J_{p,k} &= \left({}^{i}G_{j}^{p,q}\right) {}^{j}J_{q,k} \quad . \qquad (1.12) \end{aligned}$$

The shorthand notation ${}^{i}G_{j}$ is used when the transformation has no translation and the notation ${}^{i}G^{p,q}$ is used when the transformation has no rotation.

The twists that a manipulator can accomplish with joint-rate control in a given configuration are know as the *twists of freedom* [5, 8, 22],

$${}^{i}\mathcal{V}_{f} = \mathrm{Range}[{}^{i}J] \quad , \tag{1.13}$$

where $\mathcal{V}$ represents a twist manifold and i is the frame of expression. The twist of freedom manifold is a subspace.

It is important when writing vectors, matrices, and manifolds to make the frame of expression clear. In this dissertation, the expression frame, if not explicitly written as a leading superscript, will be otherwise described in the context of the discussion.

Note that throughout this dissertation, a calligraphic symbol (such as $\mathcal{V}$) represents a *manifold* (or set) of vectors or screws. Therefore, $\mathcal{X} = \{X_i\}$ is the manifold of vectors or screws X_i, for various i. The column vectors of the matrix $[\mathcal{X}]_b$ constitute a basis for $\mathcal{X}$. The matrix E_x converts the basis set, $[\mathcal{X}]_b$, to a matrix in column-reduced echelon form [56], $[\mathcal{X}]_b E_x$.

The application of a wrench W at the end-effector of a static serial manipulator will induce a balancing generalized-force vector τ_w,

$$\tau_w = J^{\tau} W \quad , \tag{1.14}$$

where a *wrench*, $W = [f^\tau,\ n^\tau]^\tau$, is the noncommensurate 6-vector composed of the two 3-vectors of forces f and moments n. A *generalized-force vector*, τ, is the n-vector of joint torques (for revolute joints) and/or joint forces (for prismatic joints).

The matrix ${}^iW_{a,p} = [{}^if_p^\tau,\ {}^in_{a,p}^\tau]^\tau$ represent a wrench at point p expressed in frame i, with the moments taken about point a. When the subscript a is omitted it is understood that the point a is at the point p, so that ${}^iW_p = {}^iW_{p,p}$. When both subscripts are omitted the origin of the frame is the point at which moments are taken, *i.e.,* ${}^iW = {}^iW_{i,i}$

Wrenches transform via the wrench coordinate transformation matrix A,

$$ {}^iW_p = ({}^iA_j^{p,q})\ {}^jW_q \quad , \tag{1.15} $$

where

$$ {}^iA_j^{p,q} = \begin{bmatrix} {}^iR_j & [0]_{3,3} \\ {}^iB_{p,q}\,{}^iR_j & {}^iR_j \end{bmatrix} \quad . \tag{1.16} $$

Equations (1.7)-(1.9) can also be used to show that

$$ \left({}^iA_j^{p,q}\right)^{-1} = {}^jA_i^{q,p} \quad \text{, and} \tag{1.17} $$

$$ \left({}^jG_i^{q,p}\right)^\tau = {}^iA_j^{p,q} \quad . \tag{1.18} $$

The wrenches applied at the end effector that require no joint forces for balancing are know as the *wrenches of constraint*, ${}^i\mathcal{W}_c$, and form a subspace,

$$ {}^i\mathcal{W}_c = \mathrm{Null}[{}^iJ^\tau] \quad . \tag{1.19} $$

These wrenches will cause no joint motion when applied to a static manipulator. Manipulators (of at least 6 joints) in configurations with Jacobian of rank 6 have no constraint wrenches, *i.e.,* some nonzero joint forces are required to balance every possible wrench.

Notice that the above twists and wrenches are screws (defined below) expressed in axis coordinates and ray coordinates [27, 30], respectively. The designations of *Plücker* *ray coordinates* and *Plücker axis coordinates* are based on the original formulation of screw theory by Ball in 1900 [5]. Ball defined lines in two ways, each independently leading to coordinate system definitions: the join of two points lead to ray coordinates and the meet of two planes lead to axis coordinates. These sets of identical but reordered coordinates are know as the homogeneous Plücker line coordinates. The distinction is only necessary when lines or screws in different Plücker coordinates are used simultaneously, as is the case with the traditional algebraic descriptions of twists and wrenches previously defined.

A screw \$ is defined as a line with an associated pitch h. For example, the motion defined by a physical screw being advanced into a pre-threaded hole can be characterized by the following screw (in axis coordinates),

$$\$^{\text{axis}} = \begin{bmatrix} hS \\ S \end{bmatrix} = \begin{bmatrix} S_0 \\ S \end{bmatrix} \text{ (in axis coords)}, \tag{1.20}$$

where the line passes through the coordinate system origin. (A more general description is given in (1.24) below.) The vector S is a commensurate 3-vector in the direction of linear motion and the rotation is about this axis using the right-hand-rule. For every θ radians of rotation, the screw advances by $h\theta$ in the S direction.

A screw may also be defined as a linear combination of unlimited lines [5, 25]. An unlimited line L is defined with two vectors: a unit vector S in the direction of the line and a vector r from the coordinate system origin to any point on the line,

$$L^{\text{axis}} = \begin{bmatrix} r \times S \\ S \end{bmatrix} = \begin{bmatrix} S_0 \\ S \end{bmatrix} \text{ (in axis coords)}. \tag{1.21}$$

Lines also have the property that $S \odot S_0 = S \odot (r \times S) = 0$. The ray coordinate version of this same line is

$$L^{\text{ray}} = \begin{bmatrix} S \\ r \times S \end{bmatrix} = \begin{bmatrix} S \\ S_0 \end{bmatrix} \text{ (in ray coords)}. \tag{1.22}$$

A linear combination of two lines in axis (ray) coordinates creates a screw in axis (ray) coordinates,

$$\gamma_{\mathrm{r}}\$_{\mathrm{r}}^{\mathrm{axis}} = \gamma_1 L_1^{\mathrm{axis}} + \gamma_2 L_2^{\mathrm{axis}} = \begin{bmatrix} \gamma_1(r_1 \times S_1) + \gamma_2(r_2 \times S_2) \\ \gamma_1 S_1 + \gamma_2 S_2 \end{bmatrix} = \gamma_{\mathrm{r}} \begin{bmatrix} S_{0\mathrm{r}} \\ S_{\mathrm{r}} \end{bmatrix} . \tag{1.23}$$

For screws, $S_{\mathrm{r}} \odot S_{0r} = h_{\mathrm{r}}$, where h_{r} the pitch of the resultant screw. Therefore screws are not lines except in the special case when the pitch is zero. The resultant screw can be written as

$$\gamma_{\mathrm{r}}\$_{\mathrm{r}}^{\mathrm{axis}} = \gamma_{\mathrm{r}} \begin{bmatrix} (r_{\mathrm{r}} \times S_{\mathrm{r}}) + h_{\mathrm{r}} S_{\mathrm{r}} \\ S_{\mathrm{r}} \end{bmatrix} . \tag{1.24}$$

The differences in equations (1.20) and (1.24) are due to different coordinate system definitions. If $r_{\mathrm{r}} = 0$, *i.e.,* the coordinate system origin is on the line of rotation, the two equations are identical. A general screw can always be converted to a "pure screw" as in (1.20) by a twist coordinate transformation for axis coordinate screws or a wrench coordinate transformation for ray coordinate screws. For example, a twist coordinate transformation will transform the pure axis coordinate screw into a general axis coordinate screw,

$$G \begin{bmatrix} h\omega \\ \omega \end{bmatrix} = \begin{bmatrix} R & BR \\ [0]_{3,3} & R \end{bmatrix} \begin{bmatrix} h\omega \\ \omega \end{bmatrix} = \begin{bmatrix} hR\omega + BR\omega \\ R\omega \end{bmatrix} . \tag{1.25}$$

Note that coordinate translations (B) do not affect the angular velocity vector—the bottom component in the right-hand-side of equation (1.25). Although rotations affect both parts of the screw, if there is no translation, the rotation will not affect the apparent purity of a screw viewed in each of the frames.

As stated above, the pitch of a screw can be found simply by

$$h = \frac{S_0 \odot S}{|S|^2} \;\; , |S| \neq 0. \tag{1.26}$$

If S is the zero vector, the screw is said to have infinite pitch and (1.25) is replaced by

$$G \begin{bmatrix} v \\ 0_3 \end{bmatrix} = \begin{bmatrix} R & BR \\ [0]_{3,3} & R \end{bmatrix} \begin{bmatrix} v \\ 0_3 \end{bmatrix} = \begin{bmatrix} Rv \\ 0_3 \end{bmatrix} . \tag{1.27}$$

Note that the translation B has no affect on the resulting screw representation. If the pitch is zero, S_0 is zero and the screw represents a pure rotation.

The translation that will move a general axis screw to a pure axis screw is

$$b = \frac{S_0 \times S}{|S|^2} \quad , |S| \neq 0. \tag{1.28}$$

where B can be found from b with (1.6).

All rigid body motion is instantaneously equivalent to a screw motion twist [9]. The twist defined previously, $V = [v_0,\ \omega]$, is equal to a linear velocity v_0 (referenced to some origin 0) and an angular velocity ω, a free vector [25] and is here defined as a screw in Plücker axis coordinates [48, 49]. A twist can also be represented in Plücker ray coordinates, $V^{\text{ray}} = [\omega,\ v_0]$.

Similarly, a wrench is instantaneously equivalent to a force and moment on a rigid body. The Plücker ray coordinates of a wrench, $W = [f,\ n_0]$, is equal to a force f in the direction of the wrench and a moment n referenced about origin 0. A wrench can also be represented in Plücker axis coordinates, $W^{\text{axis}} = [m_0,\ f]$.

Unless otherwise noted, twists will be expressed in Plücker axis coordinates and wrenches will be expressed in Plücker ray coordinates.

The matrix Δ [36] transforms a screw or line in axis coordinates to a screw or line in ray coordinates and a screw or line in ray coordinates to a screw or line in axis coordinates,

$$\$^{\text{ray}} = \Delta \$^{\text{axis}} \tag{1.29}$$

$$\$^{\text{axis}} = \Delta \$^{\text{ray}} \tag{1.30}$$

$$\Delta = \begin{bmatrix} [0]_{3,3} & I_3 \\ I_3 & [0]_{3,3} \end{bmatrix} . \tag{1.31}$$

The matrix Δ is an unitary matrix (and therefore also an orthogonal matrix) with the properties

$$\Delta = \Delta^{-1} \qquad \Delta = \Delta^{\tau} \qquad \Delta\Delta = I_6 \ . \tag{1.32}$$

The matrix Δ is an example of a more general transformation, defined as a *correlation* [27] that maps an axis screw to a ray screw (or a ray screw to an axis screw). A *collineation* maps a ray screw to a ray screw (or an axis screw to an axis screw).

The reciprocal or *Klein product* [5, 22] of any two screws in identical axis or ray coordinates—twists V_1 and V_2, for example—is defined as

$$V_1 \circ V_2 = V_1 \odot \Delta V_2 = V_1^T \Delta V_2 \tag{1.33}$$

$$= v_1 \odot \omega_2 + v_2 \odot \omega_2 \ , \tag{1.34}$$

where $\odot$ represents the Euclidean inner or dot product.

The Klein product of a screw in axis coordinates, and a screw in ray coordinates V and W is

$$V \circ W = V \odot W = V^T W = v \odot f + \omega \odot n \ . \tag{1.35}$$

Notice that no Δ matrix is needed in the expansion of the Klein product of a twist and a wrench, whereas the Δ matrix is needed in the expansion of the Klein product of two twists or two wrenches. The Klein product of a twist and wrench of the end-effector of a serial manipulator gives the *instantaneous virtual power (work)* [36] that the manipulator end-effector contributes to the environment.

A well known important characteristic of the reciprocal product is that it is invariant to coordinate transformations. This is shown in the following theorem and proof [5]. The proof is given to provide the reader an understanding of the notation and mathematics involved.

<u>*Theorem 1*</u> *The reciprocal product of a manipulator twist and wrench expressed at the same point and in the same coordinate system is invariant to coordinate transformations.*

Proof

$$\begin{aligned}
{}^{i}V_{p,q} &= \left({}^{j}G_{i}^{p,q}\right)^{-1} {}^{j}G_{i}^{p,q}\; {}^{i}V_{p,q} && (1.36)\\
&= \left({}^{i}G_{j}^{q,p}\right) {}^{j}G_{i}^{p,q}\; {}^{i}V_{p,q} \quad \text{, as shown in (1.10)} && (1.37)\\
{}^{i}V_{p,q} \circ {}^{i}W_{p} &= \left({}^{i}G_{j}^{q,p}\right) {}^{j}G_{i}^{p,q}\; {}^{i}V_{p,q} \circ {}^{i}W_{p} && (1.38)\\
&= {}^{j}G_{i}^{p,q}\; {}^{i}V_{p,q} \circ \left({}^{i}G_{j}^{q,p}\right)^{\tau} {}^{i}W_{p} && (1.39)\\
&= {}^{j}G_{i}^{p,q}\; {}^{i}V_{p,q} \circ \left({}^{j}A_{i}^{p,q}\right) {}^{i}W_{p} \quad \text{, , as shown in (1.18)} && (1.40)\\
&= {}^{j}V_{q,k} \circ {}^{j}W_{q} \quad \text{, as shown in (1.11) and (1.15).} && (1.41)
\end{aligned}$$

■

The twist or screw motion created by a single revolute joint i is a pure rotation,

$${}^{i-1}V^{\text{rev}} = \left[0,\ 0,\ 0,\ 0,\ 0,\ \dot{\theta}_i\right]^{\tau} = \begin{bmatrix} 0_3 \\ \dot{\theta}_i \hat{z} \end{bmatrix} \quad , \tag{1.42}$$

where the above equation is expressed in the frame of the previous joint $i-1$, and $\hat{z}$ is the vector $[0,\ 0,\ 1]^{\tau}$. The twist coordinate transformation matrix enables this screw to be expressed in different coordinate frames—as shown in (1.25). To express this screw in various coordinate frames, the twist coordinate transformation matrix may be employed as shown in (1.25). When the frame is translated to a frame j that is located by vector b from the frame $i-1$, the screw motion is

$$\begin{aligned}
{}^{j}V^{\text{rev}} &= \begin{bmatrix} b \times \hat{z} \\ \dot{\theta}_i \hat{z} \end{bmatrix} && (1.43)\\
&= \begin{bmatrix} B\hat{z} \\ \dot{\theta}_i \hat{z} \end{bmatrix} && (1.44)\\
&= \begin{bmatrix} [b_y,\ -b_x,\ 0]^{\tau} \\ \dot{\theta}_i \hat{z} \end{bmatrix} \quad , && (1.45)
\end{aligned}$$

where B is the screw symmetric matrix of (1.6) corresponding to the translation vector $b = [b_x,\ b_y,\ b_z]^{\tau}$.

When the frame is rotated to a frame k with no translation from frame $i-1$, the screw motion is

$${}^{k}V^{\text{rev}} = \left[0,\ 0,\ 0,\ 0,\ 0,\ \dot{\theta}_i\right]^{\tau} = \begin{bmatrix} 0_3 \\ \dot{\theta}_i\, {}^{k}R_{i-1} \hat{z} \end{bmatrix} \quad . \tag{1.46}$$

The twist or screw motion created by a single prismatic joint i is a pure translation,

$$ {}^{i-1}V^{\text{pris}} = \left[0,\ 0,\ \dot{d}_i,\ 0,\ 0,\ 0,\ \right]^{T} = \begin{bmatrix} \dot{d}_i\hat{z} \\ 0_3 \end{bmatrix} \ . \tag{1.47} $$

Again the twist coordinate transformation matrix can be employed to express this screw in various coordinate systems. An arbitrary coordinate transformation ${}^{k}G^{p,q}_{i-1}$ rotates the twist to frame k while the translation has no affect for any p and q,

$$ {}^{k}V^{\text{pris}} = {}^{k}G^{p,q}_{i-1}\ {}^{i-1}V^{\text{pris}} = \begin{bmatrix} \dot{d}_i{}^{k}R_{i-1}\hat{z} \\ 0_3 \end{bmatrix} \ . \tag{1.48} $$

That translation has no affect on this twist was verified by symbolically performing the multiplication ${}^{k}G^{p,q}_{i-1}\ {}^{i-1}V^{\text{pris}}$ in (1.27).

Screws can be added to form new screws. In this manner the motion of the end-effector (or any other point) of a serial manipulator may be found by a summation of the screws of each of the joints,

$$ \begin{aligned} V &= V_1 + V_2 + \cdots + V_n && (1.49) \\ &= \dot{q}_1\$_1 + \dot{q}_2\$_2 + \cdots + \dot{q}_n\$_n && (1.50) \\ &= [\$_1,\ \$_2,\ \cdots,\ \$_n]\,\dot{q} && (1.51) \\ &= J\dot{q}\ \ , && (1.52) \end{aligned} $$

where $\dot{q}$ is the vector of manipulator joint rates of (1.2) and (1.52) is identical to (1.1). To perform the addition of screws, it is first necessary to reference them to the same coordinate frame and point via the appropriate screw coordinate transformation matrices, *e.g.,* the summation of the screws in (1.50) is actually accomplished with the equation

$$ {}^{i}V = \sum_{j=1}^{n} \dot{q}_j\ {}^{i}G^{i,n}_{j}\ {}^{j}\$_j \ \ . \tag{1.53} $$

Any twist can be constructed by six or less independent screws each representing either a prismatic or a revolute motion. Therefore a *virtual manipulator* can always be constructed to instantaneously accomplish any twist. Griffis [25] defines a virtual manipulator as any imaginary serial manipulator "whose joint displacements and speeds uniquely describe any permissible twist (V_f) and any permissible position and orientation of its end-effector." The permissible end-effector wrenches (W_c) together with the twists completely describe the instantaneous kinematics of a real or virtual manipulator end-effector.

Theorem 2 below, given in [5], shows that the reciprocal product of any twist of freedom and any wrench of constraint must be zero. The proof is shown to give the reader an insight to the concept of reciprocity.

Theorem 2 *The Klein or reciprocal product of V_f and W_c, is zero,* i.e.,

$$V_f \circ W_c = 0 \ \ , \forall V_f \in \mathcal{V}_f \ \textit{and} \ \forall W_c \in \mathcal{W}_c \ \ . \tag{1.54}$$

Proof

$$V_f = J\dot{q} \ \ , \forall V_f \in \mathcal{V}_f \text{ and some } \{\dot{q}\} \tag{1.55}$$

$$(V_f)^{\tau} = \dot{q}^{\tau} J^{\tau} \tag{1.56}$$

$$(V_f)^{\tau} W = \dot{q}^{\tau} J^{\tau} W \ \ . \tag{1.57}$$

Now let W be a constraint wrench $W_c \in \mathcal{W}_c$, so that

$$(V_f)^{\tau} W_c = \dot{q}^{\tau} J^{\tau} W_c \ \ . \tag{1.58}$$

But $J^{\tau} W_c = 0$ by definition in (1.19), so

$$(V_f)^{\tau} W_c = 0 \ \ . \tag{1.59}$$

But by the definition of the Klein product in (1.35),

$$(V_f)^{\tau} W_c = V_f \circ W_c \ \ , \tag{1.60}$$

so that $V_f \circ W_c = 0$.

■

This means that the manipulator can do no work with any wrench of constraint or, alternatively, can not move with the screw motion of any ray coordinate constraint wrench interpreted as an axis coordinate twist.

The reciprocity relationship between $\mathcal{V}_f$ and $\mathcal{W}_c$ has been inadvertently (and inappropriately) used by researchers to characterize the entire space through the use of the direct sum decomposition of the 6-space of position and orientation.

The fundamental theorem of linear algebra [56] states that

$$\Re^m = \text{Range}[A] \oplus \text{Null}[A^\tau] \ , \tag{1.61}$$

where m is the number of rows of A. The symbol $\oplus$ represents the direct sum and implies that $\text{Range}[A] \cap \text{Null}[A^\tau] = \{0\}$ and $\Re^m = \text{Range}[A] \cup \text{Null}[A^\tau]$. Applying this theorem to robotics by letting A be the Jacobian can be misleading,

$$\Re^6 \stackrel{?}{=} \text{Range}[J] \oplus \text{Null}[J^\tau] \ . \tag{1.62}$$

Since J has physical meaning, with terms not all of the same units, the implication of this theorem applied to robotics is that the total space is a combination of axis coordinate (twists) and ray coordinate (wrenches). The subspaces $\text{Range}[J]$ and $\text{Null}[J^\tau]$ are noncommensurate. What does it mean to decompose a vector (or screw) into the sum of an axis coordinate vector and ray coordinate vector? This problem will be addressed in Chapter 6.

1.2 The Pseudo- and Generalized-Inverses

The Moore-Penrose pseudo-inverse and the weighted generalized-inverse can both be used to solve linear equations. Of course each of the solutions is based on different optimality conditions for their solutions.

1.2.1 The Moore-Penrose Pseudo-Inverse

The Moore-Penrose pseudo-inverse gives a unique minimum norm least-squares solution to a linear equation,

$$u = Ax \ , \tag{1.63}$$

for example. The pseudo-inverse of A ($A \in \Re^{(m \times n)}$), is denoted $A^\dagger$ and has the following properties [6, 34]:

$$\begin{aligned} AA^\dagger A &= A \ , & (1.64) \\ A^\dagger A A^\dagger &= A^\dagger \ , & (1.65) \\ (AA^\dagger)^\tau &= AA^\dagger \ , & (1.66) \\ (A^\dagger A)^\tau &= A^\dagger A \ . & (1.67) \end{aligned}$$

The pseudo-inverse can be found through a full-rank factorization of A, $A = FC$, where $F \in \Re^{(m \times r)}$ has full column rank r and $C \in \Re^{(r \times n)}$ has full row rank r. The pseudo-inverse of A can be expressed as

$$\begin{aligned} A^\dagger &= C^\tau (F^\tau A C^\tau)^{-1} F^\tau & (1.68) \\ &= C^\tau (CC^\tau)^{-1} (F^\tau F)^{-1} F^\tau & (1.69) \\ &= C^\dagger F^\dagger \ . & (1.70) \end{aligned}$$

The unique minimum norm least-squares solution to (1.63) is therefore

$$x_s = A^\dagger u \ . \tag{1.71}$$

The solution x_s, is a *least-squares* solution in that the residual (if any), $|u - Ax|$, is minimized, where $|\cdot|$ is the Euclidean vector norm (see equation (1.72)). The solution x_s is *minimum norm* since any other solutions x_1 to $Ax = u$ has Euclidean norm $|x_1| > |x_s|$.

A least-squares solution is obtained if (1.64) and (1.66) are true and the solution is minimum norm if (1.64) and (1.67) are true [6].

It is a fortunate fact that the least-squares solution and the minimum norm solution are identical for linear systems and equal to the pseudo-inverse solution.

The Euclidean norm of a vector $x \in \Re^n$ (also known as the square root of the Euclidean inner-product of x with itself) is defined as

$$\begin{aligned} |x| &= +\sqrt{|x|^2} \ , \\ |x|^2 &= <x, x> = x \odot x = x^\tau x = \sum_{i=1}^{n} x_i^2 \ . \end{aligned} \tag{1.72}$$

If matrix A has full row rank or full column rank, (1.68)-(1.70) has the simplified solutions

$$A^\dagger = A^\tau (AA^\tau)^{-1} \ , \ \ A \text{ full row rank, and} \tag{1.73}$$

$$A^\dagger = (A^\tau A)^{-1} A^\tau \ , \ \ A \text{ full column rank.} \tag{1.74}$$

These equations are derived directly from (1.69), substituting $F = I_r$ when A has full column rank and $C = I_r$ when A has full row rank. (Matrix I_r is the $r \times r$ identity matrix.)

1.2.2 The Weighted Generalized-Inverse

The weighted generalized-inverse gives a unique minimum M_x-norm least M_u-squares solution to a linear equation. The weighted generalized-inverse of A (called the generalized-inverse throughout the rest of this dissertation), is denoted $A^\#$ and has the following properties [6, 19]:

$$AA^\# A = A \ , \tag{1.75}$$

$$A^\# AA^\# = A^\# \ , \tag{1.76}$$

$$(M_u AA^\#)^\tau = M_u AA^\# \ , \tag{1.77}$$

$$(M_x A^\# A)^\tau = M_x A^\# A \ . \tag{1.78}$$

The matrices M_x and M_u are metrics. A metric is a symmetric positive definite matrix.

The generalized-inverse of A [6, 7, 19], with the same full-rank factorization $A = FC$ discussed previously, is

$$\begin{aligned} A^{\#} &= M_x^{-1}C^{\tau}(F^{\tau}M_u A M_x^{-1}C^{\tau})^{-1}F^{\tau}M_u && (1.79)\\ &= \left[M_x^{-1}C^{\tau}(CM_x^{-1}C^{\tau})^{-1}\right]\left[(F^{\tau}M_u F)^{-1}F^{\tau}M_u\right] && (1.80)\\ &= C^{\#}F^{\#} \ , && (1.81)\end{aligned}$$

where $F^{\#}$ and $C^{\#}$ are defined by (1.81) and the bracketed expressions in (1.80).

The unique minimum M_x-norm least M_u-squares solution to (1.63) is therefore

$$x_s = A^{\#}u \ . \qquad (1.82)$$

The solution x_s, is a *least M_u-squares* solution in that the residual (if any), $|u - Ax|_{M_u}$, is minimized, where $|\cdot|_M$ is defined below in (1.83). The solution x_s is minimum M_x-norm since any other solutions x_1 to $Ax = u$ has M_x-norm $|x_1|_{M_x} > |x_s|_{M_x}$.

The M-norm of vector a,

$$\begin{aligned} |a|_M &= +\sqrt{|a|_M^2} \ , \\ |a|_M^2 &= < a, Ma >= a \odot Ma = \sum_{i=1}^{n} a_i(Ma)_i \ . && (1.83)\end{aligned}$$

In addition to the positive definite requirement for a metric, a metric must also make the corresponding square of the M-norm physically consistent [15], *e.g.,* $a \odot Ma$ must be physically consistent for any a if M is to be considered a valid metric.

A least-M_u squares solution is obtained if (1.75) and (1.77) are true and the solution is minimum M_x-norm if (1.75) and (1.78) are true [7].

It is a fortunate fact that the least M_u-squares solution and the minimum M_x-norm solution are identical for linear systems and equal to the generalized-inverse solution given in (1.79)-(1.81).

In order for solutions to be invariant to coordinate transformations [19] in both the spaces defined by u and x, the metrics must transform via a specific congruence transformation [56],

$$M' = G^{\tau} M G \quad . \tag{1.84}$$

If $u' = G_u^{-1} u$, then the metric for u' must be $M_{u'} = G_u^{\tau} M_u G_u$. This will insure that the M_u-norm is invariant, $|u'|^2_{M_{u'}} = |u|^2_{M_u}$. The metric M_x must also transform via a congruence transformation, $M_{x'} = G_x^{\tau} M_x G_x$, where $x' = G_x^{-1} x$.

1.3 Eigenvalues, Eigenvectors and SVD

Eigenvalues and eigenvectors of an $n \times n$ matrix A are defined [34] by the equation,

$$A e^{(i)} = \lambda^{(i)} e^{(i)} \quad , \tag{1.85}$$

where the i eigenvalues and eigenvectors are represented by $\lambda^{(i)}$ and $e^{(i)}$, respectively.

The singular value decomposition (SVD) of an $m \times n$ matrix A of rank r is defined [34, 56] by the equation

$$A = U \Sigma V^{\tau} \tag{1.86}$$

where Σ is an $m \times n$ matrix with the singular values of A (σ_i) on the main diagonal, U is an $m \times m$ orthogonal matrix, and V is an $n \times n$ orthogonal matrix.

The columns of U are the eigenvectors of AA^{τ} and the columns of V are the eigenvectors of $A^{\tau}A$. The r singular values are the nonnegative square roots of the nonzero eigenvalues of both AA^{τ} and $A^{\tau}A$, *i.e.,* the eigenvalues of AA^{τ} and $A^{\tau}A$ are equal to the square of the singular values of A.

The eigenvalues are preserved for similarity transformations, $B = S^{-1}AS$, and the eigenvectors of B are $S^{-1}e^{(i)}$. Eigenvalues are not preserved under congruence transformations, $B = S^{\tau}AS$ (unless S is a rotation, in which case $S^{\tau} = S^{-1}$ and B is also a similarity transformation).

CHAPTER 2
LINEAR NONCOMMENSURATE SYSTEMS

For linear noncommensurate system, $u = Ax$, the requirements on the structure of A are determined in this section, where A is an $n \times m$ matrix, x is a noncommensurate m-vector, and u is a noncommensurate n-vector. Upon expanding $u = Ax$, it is found that

$$u_i = \sum_{j=1,m} a_{ij} x_j \quad , \tag{2.1}$$

so that

$$\text{units}[a_{ij}]\text{units}[x_j] = \text{units}[u_i] \quad . \tag{2.2}$$

Using two terms in the sum of (2.1) for two elements of u, we get

$$z_i \;=\; a_{ij} x_j + a_{ik} x_k \tag{2.3}$$

$$z_l \;=\; a_{lj} x_j + a_{lk} x_k \quad , \tag{2.4}$$

for all i, j, k, l, where $\text{units}[z_i] = \text{units}[u_i]$. Solve (2.3) for x_k and substitute the result into (2.4) to get

$$z_l - \frac{a_{lk}}{a_{ik}} z_i = (a_{ik} a_{lj} - a_{ij} a_{lk}) \frac{x_j}{a_{ik}} \tag{2.5}$$

Therefore, for physically consistency,

$$\text{units}[a_{ik}]\text{units}[a_{lj}] = \text{units}[a_{ij}]\text{units}[a_{lk}] \quad , \tag{2.6}$$

or

$$\text{units}[\frac{a_{ik}}{a_{ij}}] = \text{units}[\frac{a_{lk}}{a_{lj}}] \quad . \tag{2.7}$$

In other words, if $m-2$ columns and $n-2$ rows are eliminated, the determinant of the remaining 2×2 matrix must be physically consistent for the system to be noncommensurate.

Using three terms in the sum of (2.1) for three elements of u, we get three equations similar to (2.3) and (2.4). Solving these equations leads to a condition similar to that shown in (2.6), *i.e.*, if $m-3$ columns and $n-3$ rows are eliminated, the determinant of the remaining 3×3 matrix must be physically consistent for the system to be noncommensurate.

By induction, the above technique shows that for all $i \geq 2$, and $i \leq m \leq n$ or $i \leq n \leq m$, if $m-i$ columns and $n-i$ rows are eliminated, the determinant of the remaining $i \times i$ matrix must be physically consistent for the system to be noncommensurate.

Another requirement on matrix A is found by viewing A as a matrix of column vectors $A_{(\cdot,i)}$,

$$u = \sum_{i=1,m} x_i A_{(\cdot,i)} \quad . \tag{2.8}$$

It is evident that the units of any two columns of A must be proportional. This is an alternate way to express the results of (2.7) and a simple way to visually deduce whether or not a matrix is a candidate noncommensurate linear system matrix.

All linear systems are either commensurate, noncommensurate, or physically inconsistent. Commensurate and noncommensurate systems are physically consistent systems. For commensurate systems, all elements of the A matrix have identical units.

2.1 Eigensystem In Noncommensurate Systems

As was mentioned at the start of Section 1.1, many researchers make use of the eigenvalues, eigenvectors, or singular values of matrices whose eigenvalues and singular values are not invariant to changes in scale or coordinate transformations. These are therefore not true "eigensolutions" in the sense that they may only subjectively characterize a manipulator configuration based on a particular observer (with a choice of scale and coordinate frame of reference) as opposed to a more relevant objective characterization of a manipulator configuration.

2.2 Conditions for Physically Consistent Eigensystems

When does a matrix A have physically consistent eigenvalues and eigenvectors? Let A be an $n \times n$ matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} , \tag{2.9}$$

and let the domain of A be $\mathcal{X}^n$, where $\mathcal{X}^n$ is a space with physical units. The $\mathcal{X}^n$-space can be characterized as follows. Let β be an n-vector of possibly distinct physical units

$$\beta = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix}^{\tau} . \tag{2.10}$$

Any $x \in \mathcal{X}^n$ is equivalent to an item-wise multiplication of β and y, $y \in \Re^n$, *i.e.*,

$$x = \beta \otimes y = \begin{bmatrix} \beta_1 y_1 & \beta_2 y_2 & \cdots & \beta_n y_n \end{bmatrix}^{\tau} \tag{2.11}$$

$$x \in \mathcal{X}^n \tag{2.12}$$

$$y \in \Re^n , \tag{2.13}$$

so that $\Re^n \overset{\beta}{\mapsto} \mathcal{X}^n$ and each eigenvector of A from (1.85), $e^{(i)}$, is an element of $\mathcal{X}^n$-space.

Substituting (2.9) into (1.85) and performing the matrix multiplication results in the following equations:

$$\begin{array}{lcl} a_{11}{e_1}^{(i)} + a_{12}{e_2}^{(i)} + \cdots + a_{1n}{e_n}^{(i)} & = & \lambda^{(i)}{e_1}^{(i)} \\ a_{21}{e_1}^{(i)} + a_{22}{e_2}^{(i)} + \cdots + a_{2n}{e_n}^{(i)} & = & \lambda^{(i)}{e_2}^{(i)} \\ \vdots & & \vdots \\ a_{n1}{e_1}^{(i)} + a_{n2}{e_2}^{(i)} + \cdots + a_{nn}{e_n}^{(i)} & = & \lambda^{(i)}{e_n}^{(i)} \end{array} . \tag{2.14}$$

Recognizing that only quantities with identical physical units may be added leads to the following theorem.

<u>*Theorem 3*</u> *The equation $Ax = \lambda x$ is physically consistent if and only if*

$\text{units}[a_{kj}]\text{units}[x_j] = \text{units}[\lambda]\text{units}[x_k]$, *for all j and k.*

Proof

By hypothesis, $\sum_{j=1}^{n} a_{kj}x_j = \lambda x_k$ for all k. Recognizing that only identical physical units can be added together, we immediately conclude that units$[a_{kj}]$units$[x_j]$ = units$[\lambda]$units$[x_k]$, for all j and k.

Now, assume units$[a_{kj}]$units$[x_j]$ = units$[\lambda]$units$[x_k]$ for all j and k. Clearly, the equation $\sum_{j=1}^{n} a_{kj}x_j = \lambda x_k$ is physically consistent for all k, *i.e.,* $Ax = \lambda x$ is physically consistent.

■

Observe that units$[a_{kj}]$units$[x_j]$ = units$[\lambda]$units$[x_k]$ implies that units$[\lambda]$ = units$[a_{ii}]$, for all i. Hence, any matrix with a physically consistent eigenvalue equation must have diagonal elements with the same physical units and all its eigenvalues must have those same units.

A simple test for a physically consistent eigensystem is the validity of the below equation for each element in matrix A,

$$\text{units}[a_{kj}]\text{units}[a_{jk}] = \text{units}[a_{ii}^2] \quad . \tag{2.15}$$

Since the singular values of A are the nonnegative square roots of the nonzero eigenvalues of both AA^τ and $A^\tau A$, a test on these matrix products (similar to the tests discussed above for the eigensystem of A) will determine if the SVD of A is physically consistent. The conditions for the physical consistency of the SVD of A are stated in Corollary 1 below.

Corollary 1 The singular value decomposition of A, $A = U\Sigma V^\tau$, is physically consistent if and only if units$[b_{kj}]$units$[x_j]$ = units$[\lambda]$units$[x_k]$*, for all j and k, where $B = AA^\tau$ or $B = A^\tau A$, and $Bx = \lambda x$.*

Proof

This follows directly from Theorem 3 and the properties of SVD, *i.e.,* the eigensystem tests on the matrices AA^τ and $A^\tau A$ determine the singular values and orthogonal matrices U and V^τ. Therefore the test of Theorem 3 and (2.15) can be directly applied to AA^τ and $A^\tau A$ to determine if the SVD of A is physically consistent.

■

Let j in Corollary 1 be equal to k. Then units$[b_{kk}]$ = units$[\lambda]$, and all diagonal elements of B must have the same physical units. If A is an $n \times m$ matrix, then each diagonal term of B is

$$b_{kk} = \begin{cases} \sum_{j=1}^{m} a_{kj} & \text{, for } B = AA^\tau \\ \quad \text{or} & \\ \sum_{j=1}^{n} a_{jk} & \text{, for } B = A^\tau A \end{cases} \quad \text{, for all } k. \tag{2.16}$$

Therefore, all the elements in the k-th row or k-column A must have identical units, for $B = AA^\tau$ or $B = A^\tau A$, respectively. But since units$[b_{kk}]$ = units$[b_{jj}]$ for all j and k, all the elements of A must have the same units. Therefore, singular value decomposition is only valid for commensurate systems, *i.e.,*

Theorem 4 *Noncommensurate system never have a physically consistent singular value decomposition.*

The major results of this chapter are summarized below. The requirements on A for all physically consistent linear noncommensurate systems, $u = Ax$, were given in (2.6) and (2.7). The requirements for A to have a physically consistent eigensystem were given in (2.15). And, finally, it was shown that physically consistent linear noncommensurate systems do not have a physically consistent singular value decomposition. Only commensurate systems have a physically consistent SVD.

CHAPTER 3
PHYSICAL CONSISTENCY OF JACOBIAN FUNCTIONS

The manipulator Jacobian is used by many researchers in ways which result in physically inconsistent results. Several of these will be discussed in this chapter.

3.1 Inappropriate Uses of the Euclidean Norm in Robotics

A multitude of researchers [3, 31, 45, 59, 60] have characterized a robot configuration or condition in terms of the scalar quantity of the Euclidean norm. This will be shown to be invalid, in general. One or more non-Euclidean metrics are often necessary [14, 17, 19, 53, 54] to find a physically consistent (non-Euclidean) norm. Although this may not seem obvious at first glance, consider the following examples.

The twist vector V—defined in (1.3)—is composed of the translational velocity vector v and the angular velocity vector ω. The square of the Euclidean norm is often inappropriately applied to the twist vector,

$$|V|^2 = V \odot V = V^\tau \; V \quad . \tag{3.1}$$

But the expression $V \odot V$ is physically inconsistent since

$$|V|^2 \stackrel{?}{=} v \odot v + \omega \odot \omega \quad , \tag{3.2}$$

and v has units of [length/time] while ω has units of [angle/time]. This is like adding apples to oranges, generally inappropriate without a metric on the worth of an apple compared to an orange ([length/time] compared to [angle/time]).

For example, if $v^\tau = [1 \;\; 1 \;\; 1]\frac{\text{cm}}{\text{s}}$, and $\omega^\tau = [1 \;\; 1 \;\; 1]\frac{\text{rad}}{\text{s}}$, then $|V|^2 \stackrel{?}{=} 6$. Changing the scale from cm to mm will change the result to $|V|^2 \stackrel{?}{=} 303 \neq 6$!

If we define a metric for twists, M_v, we can use the M_v-norm —defined in (1.83)— to get a measure of twists, $|V|^2_{M_v} = V \odot M_v V$. The metric M_v must be positive definite and make $|V|^2_{M_v}$ physically consistent. A metric M_v can be selected such that this norm describes the kinetic energy, K, of a rigid body,

$$K = \frac{1}{2} V \odot M_v V = \frac{1}{2} V^\tau M_v V \tag{3.3}$$

The kinetic energy metric expressed at the center-of-mass with axes aligned with the body's principal axes is the *principal mass-inertia matrix* of a rigid body,

$$M_{KE} = \begin{bmatrix} m_b I_3 & [0]_{3,3} \\ [0]_{3,3} & I_b \end{bmatrix} \ , \tag{3.4}$$

where m_b is the body's mass and I_b is the body's inertia tensor at the center-of-mass expressed in principal coordinates—a diagonal matrix. We must express this metric in the same frame as the twists— see (1.84)—(or express the twists at the body's center-of-mass aligned with the body's principal axis). Transforming the metric M_{KE} to the frame of expression of the twist results in the metric

$$M_v = G_v^\tau M_{KE} G_v = \begin{bmatrix} m_b I_3 & m_b R^\tau B R \\ m_b R^\tau B^\tau R & R^\tau (I_b + m_b B^\tau B) R \end{bmatrix} \ , \tag{3.5}$$

where G_v is defined in (1.4), with $R = {}^iR_p$, $B = {}^iB_{p,i}$, i is the expression frame for the twist $V = {}^iV$, and p is the frame of the principal axes of the body. The lower right 3×3 matrix in (3.5) is the inertia matrix of the body in the twist frame. If there is no rotation between the twist frame and the principal frame, the inertia matrix is $I_b' = (I_b + m_b B^\tau B)$. This inertia matrix could have also been determined using the parallel axis theorem [42].

The metric of (3.5) is the *twist inertia matrix* of a rigid body composed of the zero-order mass-moment (mass), the first-order mass-moment (momentum), and the second-order mass-moment (inertia).

For a second example of the problem of using Euclidean norms in robotic applications, let us look at the generalized-force vector τ of the manipulator joints. The square of the Euclidean norm of τ is

$$|\tau|^2 \stackrel{?}{=} {\tau_1}^2 + {\tau_2}^2 + \cdots + {\tau_n}^2 \ . \tag{3.6}$$

If all the joints of the manipulator are revolute or all are prismatic, (3.6) is physically consistent (but this measure of the sum of the square of joint torques is probably of little value since the driving component of some joints is generally quite different from other joints). But, if the manipulator has both revolute and prismatic joints — *i.e.,* the manipulator joints angles or velocities form a noncommensurate vector—this equation sums physically inconsistent force-squared and moment-squared terms.

Let us view the Euclidean norms of V and τ from a different perspective — namely, by looking at the manipulator Jacobian defined in (1.1). Of course if the manipulator has 6 joints and J has full rank, then J^{-1} can be found and the solution to (1.1) is

$$\dot{q}_s = J^{-1}\ V \quad . \tag{3.7}$$

To solve for $\dot{q}$ when J is not a square matrix, many researchers use the pseudo-inverse $J^{\dagger}$—see (1.68)-(1.70)—and the equation

$$\dot{q}_s \stackrel{?}{=} J^{\dagger}\ V \quad . \tag{3.8}$$

For a full row rank matrix J, the pseudo inverse is

$$J^{\dagger} = J^{\tau}(JJ^{\tau})^{-1} \quad , J \text{ full row rank.} \tag{3.9}$$

Equation (3.9) is often used with redundant manipulators (manipulators with more than 6 joints). For manipulators with less than 6 joints, the pseudo-inverse for a full column rank matrix is often used,

$$J^{\dagger} = (J^{\tau}J)^{-1}J^{\tau} \quad , J \text{ full column rank.} \tag{3.10}$$

Note that the pseudo-inverse in one case involves the term JJ^{τ} and in the other case involves the term $J^{\tau}J$. There is often a units problem (physical inconsistency) with both of these terms. One of these terms also appears in each of the norm of a twist V and the norm of the generalized-force vector τ_w, as will be shown below.

From equations (3.1) and (1.1), we get the Euclidean norm of twist V of

$$|V|^2 \stackrel{?}{=} V^\tau \ V = (J\ \dot{q})^\tau (J\ \dot{q}) = \dot{q}^\tau \ (J^\tau J)\ \dot{q} \ . \tag{3.11}$$

A similar technique will be used to find an alternate form of the Euclidean norm of the generalized-force vector.

The static wrench defined in (1.14) is repeated here for convenience, $\tau_w = J^\tau W$, where τ_w is the n-vector of generalized-forces—joint torques (for revolute joints) and/or joint forces (for prismatic joints)—induced by an end-effector wrench W, and J is the manipulator Jacobian. A wrench $W = [f^\tau,\ n^\tau]^\tau$ is composed of the 3-vectors of force f and moment n.

The term JJ^τ again appears in the Euclidean norm of τ_w. Equation (3.6) can be rewritten using (1.14) as

$$|\tau_w|^2 \stackrel{?}{=} {\tau_w}^\tau \ \tau_w = W^\tau \ (JJ^\tau)\ W \ . \tag{3.12}$$

Let us now look at the physical consistency of these Euclidean norms by performing a units analysis on $J^\tau J$ and JJ^τ.

The units of a manipulator Jacobian matrix is found simply by noting that the units of the range of J is equal to the units of V and is not dependent on the structure of the manipulator. Therefore the units of elements in a Jacobian column have one of the following two forms [13, 16, 53]:

- If manipulator joint i is revolute, the i-th column of the Jacobian has the units

$$\text{units}[J_{(\cdot,i)}] = \begin{bmatrix} [L]_{3,1} \\ [U]_{3,1} \end{bmatrix} \ , \ \text{for revolute joints} \ . \tag{3.13}$$

- If manipulator joint i is prismatic, the i-th column of the Jacobian has the units

$$\text{units}[J_{(\cdot,i)}] = \begin{bmatrix} [U]_{3,1} \\ [0]_{3,1} \end{bmatrix} \ , \ \text{for prismatic joints} \ . \tag{3.14}$$

The $[\cdot]_{j,k}$ in the above equations corresponds to a $j \times k$ matrix whose elements have units of L for units of length or U for unitless. The $[0]_{j,k}$ term identifies a matrix whose elements are equal to zero (and says nothing about the elements' units).

3.2 Physical Consistency of $J^\tau J$ and JJ^τ

If all n joints manipulator are revolute, the units of $J^\tau J$ is

$$\text{units}[J^\tau J] \stackrel{?}{=} \Big[\ [L^2 + U]_{n,n}\ \Big]\ \ ,\ \ \text{for } n \text{ revolute joints,} \tag{3.15}$$

i.e., each term sums a length-squared term with a unitless term. Since the Euclidean norm of V in (3.11) requires the product $(J^\tau J)$, the Euclidean norm of V is obviously physically inconsistent, as shown in (3.2).

For noncommensurate manipulators, if the i-th and j-th joints of a manipulator are revolute, then the (i,j)-th element of the matrix $J^\tau J$ is physically inconsistent with units of

$$\text{units}[(J^\tau J)_{(i,j)}] \stackrel{?}{=} L^2 + U\ \ ,\ \text{for } i\text{-th and } j\text{-th joints revolute.} \tag{3.16}$$

If the i-th joint is revolute and the j-th joint is prismatic, then the (i,j)-th element of the matrix $J^\tau J$ is physically consistent with units of

$$\text{units}[(J^\tau J)_{(i,j)}] = L\ \ ,\ \text{for } i\text{-th joint revolute, } j\text{-th joints prismatic.} \tag{3.17}$$

If the i-th and j-th joints are both prismatic, then the (i,j)-th element of the matrix $J^\tau J$ is physically consistent with units of

$$\text{units}[(J^\tau J)_{(i,j)}] = U\ \ ,\ \text{for } i\text{-th and } j\text{-th joints prismatic.} \tag{3.18}$$

Similarly, the Euclidean norm of $\dot{q}$ is also physically inconsistent for noncommensurate manipulators, *i.e.,*

$$|\dot{q}|^2 \stackrel{?}{=} \dot{q}_1^2 + \dot{q}_2^2 + \cdots + \dot{q}_n^2\ \ , \tag{3.19}$$

making a noncommensurate vector of joint rates with units of $(L^2 + U)/T^2$, where T represents time units.

The Euclidean norm of V and the matrix $J^\tau J$ are physically consistent for an all prismatic-jointed manipulator since the entire $J^\tau J$ matrix is unitless and $V = [v^\tau,\ 0,\ 0,\ 0]^\tau$, *i.e.,* the angular velocity is zero.

Table 3.1. D-H parameters for GE P50 manipulator.

Joint Type	d	a	θ	α
R	0	0	θ_1	$\pi/2$
R	a_2	0	θ_2	0
R	a_3	0	θ_3	0
R	0	0	θ_4	$\pi/2$
R	0	0	θ_5	0

The General Electric P50 manipulator (with 5 revolute joints) has Denavit-Hartenberg parameters given in Table 3.1 and a frame 2 Jacobian

$$ {}^2J = \begin{bmatrix} 0 & 0 & 0 & a_3 s_3 & 0 \\ 0 & a_2 & 0 & -a_3 c_3 & 0 \\ -a_2 c_2 & 0 & 0 & 0 & -a_3 c_4 \\ s_2 & 0 & 0 & 0 & s_{3+4} \\ c_2 & 0 & 0 & 0 & -c_{3+4} \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} . \tag{3.20} $$

The matrix ${}^2J^\tau\, {}^2J \stackrel{def}{=} {}^2(J^\tau J)$ has elements with inconsistent physical units such as the $(4,4)$ term whose calculated value is $1 + a_3^2$.

The determinant of ${}^2(J^\tau J)$ for the P50 manipulator has terms that sum elements with units of L^4 with L^6. The determinant of $J^\tau J$ for a variety of manipulators was calculated in various frames and generally found to be physically inconsistent. A summary appears in Table 3.2. This table also shows the units of the determinant for each of the manipulators in various frames. (Refer to Appendix A for the D-H parameters for each manipulator in this table. This appendix also has the Jacobian and the determinant of $J^\tau J$ in a particular frame or frames for each of the manipulators.) The frame "general" corresponds to any nonzero translation. Pure rotations have no affect on the value of $J^\tau J$ since

$$ (J')^\tau J' = (GJ)^\tau (GJ) \tag{3.21} $$

$$ = J^\tau G^\tau G J \tag{3.22} $$

$$ G^\tau = G^{-1} \text{ , for rotations (no translation),} \tag{3.23} $$

$$ \Rightarrow (J')^\tau J' = J^\tau J \text{ , for rotations (no translation).} \tag{3.24} $$

Table 3.2. Physical units of $\mathrm{Det}[J^\tau J]$ for various non-redundant manipulators.

Manipulator Description	Coordinate Frame	Units of $\mathrm{Det}[J^\tau J]$
PR Virtual	0,1,2	U
PR Virtual	general	$U + L^2$
Planar RRR	All	L^4
Non-planar RRR	0,1,2,general	$U + L^2 + L^4$
General RRR	0,1,2,general	$U + L^2 + L^4$
PPP Orthogonal	All	U
SAR (PRP)	0,1,2	U
SAR (PRP)	3,general	$U + L^2$
RPR	0,1,2,3	$U + L^2$
RPR	general	$U + L^2 + L^4$
SCARA (RRRP)	Any	L^4
RRRP-2	0	L^2
RRRP-2	1,2,3,4,general	$L^2 + L^4$
RRRP-3	0,1	$U + L^2$
RRRP-3	2,3,4,general	$U + L^2 + L^4$
P50 (5R)	0,1,2,3,4,5	$L^4 + L^6$
P50 (5R)	t	L^4
6-jointed, $\mathrm{Det}[J] \neq 0$	Any frame	L^{6-2p}

Although the physical consistency of $J^\tau J$ assures the physical consistency of the determinant of $J^\tau J$, the inverse of this statement is not always true. For instance, the RRRP-2 manipulator in frame 0 has physically inconsistent terms in ${}^0(J^\tau J)$, but $\mathrm{Det}[{}^0(J^\tau J)] = a_2^2 S_3^2$ is physically consistent.

It will be shown in Section 6.6 that the physical consistency of the determinant of $J^\tau J$ assures that $J^\dagger$ is physically consistent.

Frames in which $J^\dagger$ is physically consistent are called decoupled frames. The reason these frames are called decoupled frames will be made clear in Chapter 6.

<u>*Definition 1*</u> A frame is called a *decouple frame* of a manipulator if the pseudo-inverse of the manipulator Jacobian in this frame is physically consistent.

The determinant of J for a manipulator with six joints can always be calculated since J is 6×6 for these robots. The physical dimensions of $\mathrm{Det}[J]$ (always physically consistent) is L^{3-p}, where p is the number of prismatic joints up to three. (Any more than three prismatic joints will mean the manipulator always has $\mathrm{Det}[J] = 0$.) The determinants of $J^\tau J$ and JJ^τ therefore have physical dimensions $L^{2(3-p)}$ and are equal since

$$\mathrm{Det}[A]\mathrm{Det}[B] = \mathrm{Det}[AB] \tag{3.25}$$

for all square matrices A and B with identical matrix dimensions. Equation (3.25) also guarantees the equality $\mathrm{Det}[J^\tau J] = \mathrm{Det}[JJ^\tau] = (\mathrm{Det}[J])^2$.

The determinant of $J^\tau J$ is zero for manipulators with more than six, joints since $J^\tau J$ can have at most rank 6, the maximum rank of J (not rank n). So instead we look at the matrix JJ^τ for redundant manipulators.

The units of JJ^τ for an all revolute joint manipulator is

$$\mathrm{units}[JJ^\tau] = \begin{bmatrix} [L^2]_{3,3} & [L]_{3,3} \\ [L]_{3,3} & [U]_{3,3} \end{bmatrix} \quad , \quad \text{for all revolute joints.} \tag{3.26}$$

The units of this matrix are physically consistent, as is the case for an all prismatic-jointed manipulator where

$$\mathrm{units}[JJ^\tau] = \begin{bmatrix} [U]_{3,3} & [0]_{3,3} \\ [0]_{3,3} & [0]_{3,3} \end{bmatrix} \quad , \quad \text{for all prismatic joints.} \tag{3.27}$$

For a noncommensurate manipulator, the JJ^τ units matrix of

$$\mathrm{units}[JJ^\tau] \stackrel{?}{=} \begin{bmatrix} [L^2+U]_{3,3} & [L]_{3,3} \\ [L]_{3,3} & [U]_{3,3} \end{bmatrix} \quad , \quad \text{for noncommensurate manipulator,} \tag{3.28}$$

is physically inconsistent.

The determinant of JJ^τ is frame independent (*i.e.,* invariant to both rotations and translations) since for $J' = GJ$,

$$\begin{aligned} \mathrm{Det}[J'(J')^\tau] = \mathrm{Det}[GJ(GJ)^\tau] &= \mathrm{Det}[GJJ^\tau G^\tau] && (3.29) \\ &= \mathrm{Det}[G]\,\mathrm{Det}[JJ^\tau]\,\mathrm{Det}[G^\tau] && (3.30) \\ &= \mathrm{Det}[JJ^\tau] \ , && (3.31) \end{aligned}$$

and the determinant of the twist coordinate transformation matrix G is one.

Table 3.3. Physical units of Det$[JJ^\tau]$ for various redundant manipulators.

Manipulator Description	Coordinate Frame	Units of Det$[JJ^\tau]$
6-jointed, Det$[J] \neq 0$	Any frame	L^{6-2p}
Anthropomorphic Arm (7R)	Any	L^6
Puma-260 +1 (7R)	Any	L^6
CESAR (7R)	Any	L^6
K-1207 (7R)	Any	L^6
3P-4R	Any	U
GP66 +1 (2R-P-4R)	Any	$L^4 + L^6$

(The determinant of JJ^τ for manipulators with less than six joints is of course zero since the rank of J and thus the rank of JJ^τ is less than six for these robots.)

The determinant of JJ^τ for several redundant manipulators was calculated and the physical consistency of the determinants corresponded to the physical consistency discussed above for the matrix JJ^τ in all cases. Table 3.3 shows the units of the determinant for each of the manipulators. See Appendix A for the Denavit-Hartenberg parameters of each of these manipulators, the Jacobian in a particular midframe, and the determinant of JJ^τ in this frame.

3.2.1 Consistency of $|u = Ax|$

A generalization of some of the above results for the physical consistency of the Euclidean norm will be shown in this section. For a linear set of equations $u = Ax$, Theorem 5 and Corollary 2 (both below) show that the physical consistency (or inconsistency) of the Euclidean norm of u can be determined by the physical consistency (or inconsistency) of $A^\tau A$.

Theorem 5 *If $u = Ax$, where A is an $m \times n$ matrix ($m \geq n$), then the for the following statements S1 through S3, S1 implies S2 and S2 implies S3, so that S1 implies S3.*

S1 The equation $|u|^2 = u \odot u = u^\tau u$ is physically consistent (inconsistent).

S2 The nonzero elements in a given column of A have identical units (not all identical units), i.e.,

$$\textit{If } a_{ik} \neq 0 \textit{ and } a_{jk} \neq 0, \textit{ then } \mathrm{units}[a_{ik}] = \mathrm{units}[a_{jk}],$$
$$\textit{for } k \in \{1, 2, \ldots, n\} \textit{ and } i, j \in \{1, 2, \ldots, m\}. \tag{3.32}$$

S3 The matrix $A^\tau A$ is physically consistent (inconsistent).

In other words, Theorem 5 tells us that the physical consistency of the Euclidean norm of u implies that all elements in a given column of A have identical units (or are equal to zero) and that $A^\tau A$ is physically consistent.

Proof

This proof is split up into two parts: the first proof shows that S1 implies S2; the second proof shows that S2 implies S3. Then by transitivity, S1 implies S3.

The following hold throughout these proofs: $i, j \in \{1, 2, \ldots, m\}$ and $k, h \in \{1, 2, \ldots, n\}$.

- Assume S1 to prove S2.
  - Since $u^\tau u$ is physically consistent, $\mathrm{units}[u_i] = \mathrm{units}[u_j] = \mathrm{units}[u]$.
  - Since $u = Ax$, $u_i = \sum_{k=1}^{n} a_{ik} x_k$.
  - Since u_i is physically consistent, $\mathrm{units}[a_{ik} x_k] = \mathrm{units}[a_{ih} x_h]$.
  - Since $\mathrm{units}[u_i] = \mathrm{units}[u_j]$, $\mathrm{units}[\sum_{k=1}^{n} a_{ik} x_k] = \mathrm{units}[\sum_{k=1}^{n} a_{jk} x_k]$.
  - But $\mathrm{units}[\sum_{k=1}^{n} a_{ik} x_k] = \mathrm{units}[a_{ik} x_k]$, so that $\mathrm{units}[a_{ik} x_k] = \mathrm{units}[a_{jk} x_k]$.
  - Therefore, $\mathrm{units}[a_{ik}] = \mathrm{units}[a_{jk}]$ and all terms in column k of A have identical units. This proves S2 given S1.

- Assume S2 to prove S3.
  - Given that units$[a_{ik}]$ = units$[a_{jk}]$.
  - Let $B = A^\tau A$, so that $b_{hk} = \sum_{i=1}^{m} a_{ih}a_{ik}$.
  - Since all elements in a column k of A are identical (units$[a_{ik}]$ = units$[a_{jk}]$), units$[b_{hk}]$ = units$[a_{ih}a_{ik}]$ so that each element b_{hk} of $B = A^\tau A$ is physically consistent. This proves S3 given S2.

■

Corollary 2 below follows directly from the above theorem when the Euclidean norm of x is physically consistent.

Corollary 2 *If $u = Ax$, where A is an $m \times n$ matrix $(m \geq n)$, and the Euclidean norm of x is physically consistent, then the three statements in Theorem 5 are equivalent and are equivalent to the statement*

S4 All elements of A must have (must not have) identical units.

Proof

To prove the corollary, it is only necessary to show that with the added condition of a physically consistent $|x|$, statement S3 of Theorem 5 implies S4 of the corollary and S4 implies S1 of the theorem.

Throughout this corollary, let $i, j \in \{1, 2, \ldots, m\}$ and $k, h \in \{1, 2, \ldots, n\}$.

- Assume $x^\tau x$, and $A^\tau A$ are physically consistent.
- Since $u = Ax$, $u_i = \sum_{k=1}^{n} a_{ik}x_k$.
- Since $x^\tau x$ is physically consistent, units$[x_k]$ = units$[x_h]$ = units$[x]$.
- Then units$[u_i]$ = units$[a_{ik}x]$ = units$[a_{ih}x]$, and units$[a_{ik}]$ = units$[a_{ih}]$. This means that all elements in the i-th row of A have identical units.

- The diagonal elements of $B = A^\tau A$ are $b_{kk} = \sum_{i=1}^{m} a_{ik} a_{ik}$. Since B is physically consistent, units$[a_{ik}]$ = units$[a_{jk}]$. This means that all elements in the k-th column of A have identical units.

- Since all elements in any row or any column of A have identical units, then all elements of A have identical units. This proves statement S4.

- Finally, I will show that statement S4 implies S1. Since the elements of x have identical units and S4 tells us that the elements of A have identical units, then the equation $u = Ax$ forces the elements of u to have identical units. Therefore, u has a physically consistent Euclidean norm. This proves statement S1.

■

A theorem similar to Theorem 6 (offered without proof) can be constructed with the following conditions relating the physical consistency of $|x|^2$, the units of all elements in each row of A, and the physical consistency of AA^τ.

Theorem 6 *If $u = Ax$, where A is an $m \times n$ matrix ($m \leq n$), then the for the following statements S1 through S3, S1 implies S2 and S2 implies S3, so that S1 implies S3.*

S1 The equation $|x|^2 = x \odot x = x^\tau x$ is physically consistent (inconsistent).

S2 The nonzero elements in a given row of A have identical units (not all identical units), i.e.,

$$\begin{aligned} &\textit{If } a_{ki} \neq 0 \textit{ and } a_{kj} \neq 0, \textit{ then } \mathrm{units}[a_{ki}] = \mathrm{units}[a_{kj}], \\ &\quad \textit{for } k \in \{1, 2, \ldots, m\} \textit{ and } i, j \in \{1, 2, \ldots, n\}. \end{aligned} \tag{3.33}$$

S3 The matrix AA^τ is physically consistent (inconsistent).

In other words, Theorem 6 tells us that the physical consistency of the Euclidean norm of x implies that all elements in a given row of A have identical units (or are equal to zero) and that AA^τ is physically consistent.

Corollary 3 follows directly from the above theorem when the Euclidean norm of u is physically consistent (and is also offered without proof).

<u>*Corollary 3*</u> *If $u = Ax$, where A is an $m \times n$ matrix ($m \leq n$), and the Euclidean norm of u is physically consistent, then the three statements in Theorem 6 are equivalent and are equivalent to the statement*

S4 All elements of A must have (must not have) identical units.

The implications of these two theorems and two corollaries are that noncommensurate systems generally need be dealt with in a more considered manner than commensurate systems which has often not been the case in robotics. Since the matrices $A^\tau A$ and AA^τ are used in the pseudo-inverse solution $x_s = A^\dagger u$, for full column rank A or full row rank A, respectively, the above theorems can be used to determine the general validity of these results. (The validity is not *absolutely* determined by the physical consistencies of these matrix products as was evidenced in the fact that the RRRP-2 has a physically inconsistent ${}^0(J^\tau J)$ but a physically consistent $\mathrm{Det}[{}^0(J^\tau J)]$ and ${}^0J^\dagger$.)

In the robotics inverse velocity problem, solving $V = J\dot{q}$ for $\dot{q}$, given V, through use of the pseudo-inverse gives physically inconsistent results due to the non-Euclidean nature of the twist and (sometimes) joint spaces. This physical inconsistency is apparent in the physical inconsistency of $J^\tau J$ or JJ^τ.

<u>3.2.2 Invalid use of Eigensystem and SVD of JJ^τ</u>

Since the pseudo-inverse for redundant manipulators of equation (3.9) contains the matrix JJ^τ, many researchers have used this factor in solving (1.1) for the joint rates or to characterize a manipulator configuration [2, 3, 12, 23, 29, 32, 39, 44, 45, 52, 57, 59, 60]. Yoshikawa [59, 60], for example, was the first of many to use $\sqrt{\mathrm{Det}(JJ^{\tau})}$ as a manipulability measure for a manipulator in a given configuration. Further, Yoshikawa (and others including [31, 46]) defined a manipulability ellipsoid with principal axes in the direction of the eigenvectors of JJ^{τ}. Each ellipsoid axis was given the length of $\sqrt{1/\lambda^{(i)}}$, where $\lambda^{(i)}$ is an "eigenvalue" of JJ^{τ}.

Recall that Theorem 3 in Section 2.2 gives the requirements for meaningful eigenvalues and eigenvectors. Even though JJ^{τ} is physically consistent for an all revolute joint manipulator (see the units matrix of 3.26), this matrix does not have an invariant eigensystem since (2.15) requires that the units of each term on the main diagonal of the matrix must be identical where in fact they are $[L^2, L^2, L^2, U, U, U]$.

The matrix JJ^{τ} for most noncommensurate manipulators also does not have meaningful eigensystems since the matrix is itself physically inconsistent. An exception to the general physical inconsistency of JJ^{τ} for noncommensurate manipulators occurs with the 3P-4R Redundant Spherical Wrist Robot with D-H parameters given in Table A.18 when expressed in a particular set of frames.

The matrix JJ^{τ} for the 3P-4R manipulator is generally physically inconsistent as expected. But in any frame with origin located at the center of the spherical wrist (the origin of frames 4, 5, 6, and 7), the matrix JJ^{τ} is physically consistent and unitless. The eigenvalues and eigenvectors of JJ^{τ} are therefore well defined by the rules given in Theorem 3 and (2.15) and are dimensionless. The eigenvalues are $[1, 1, 1, 2, 0.873, 1.912]$ and are invariant to rotation of the frame (with this origin). The invariance of eigenvalues to rotations can be deduced from the well known theorem that similarity transformations preserve eigenvalues, *i.e.,* if $Ae = \lambda e$, then $SAS^{-1}e' = \lambda e'$ for full rank S. The twist coordinate transformation matrix G acts like S in the similarity transformation derived below:

$$
\begin{aligned}
JJ^{\tau}e &= \lambda e \\
GJJ^{\tau}e &= \lambda Ge \\
e &= G^{\tau}e' \\
GJJ^{\tau}G^{\tau}e' &= \lambda GG^{\tau}e' && (3.34) \\
G^{\tau} &= G^{-1} \text{ , for rotations (no translation),} \\
GJJ^{\tau}G^{-1}e' &= \lambda GG^{-1}e' && (3.35) \\
J' &= GJ \\
J'(J')^{\tau}e' &= \lambda e' \\
&\Rightarrow \lambda \text{ invariant to rotations.} && (3.36)
\end{aligned}
$$

Notice that if translations are allowed, the congruence transformations of (3.34) results. Since $GG^{\tau} \neq I_6$, translations (and congruence transformations) do not preserve eigenvalues.

Even though JJ^{τ} for the 3P-4R manipulator in frames located at the intersection of the spherical joint axes appears to have physically meaningful eigenvalues and eigenvectors, the interpretation of this manipulability ellipsoid is problematic since the eigenvectors appears to be unitless (not the necessary wrenches that should be expected for the wrench manipulability ellipsoid discussed in Chapter 5). Moreover, as was stated in Theorem 4, noncommensurate systems never have a physically consistent SVD.

The matrix JJ^{τ} for an all prismatic-jointed manipulator (with at most three degrees-of-freedom and no orientation capabilities) also has a meaningful eigensystem but these limited manipulators will not be discussed.

Therefore, since JJ^{τ} does not have eigenvalues or eigenvectors (except for the special cases mentioned above), the above configuration characterization theory is invalid. (Several of the commonly used manipulability ellipsoids are shown in [17] to be physically inconsistent.)

It will be shown later, in Section 5, that the use of metrics on the appropriate noncommensurate twist and joint spaces (discussed in the next chapter) does not change the fact that the manipulability ellipsoid theory violates the eigensystem and SVD theorems of Section 2.2.

CHAPTER 4
INVERSE VELOCITY KINEMATICS

Several authors [14, 19, 35, 53, 54] have discussed the inappropriateness of using the pseudo-inverse in solving for the joint rates given a desired twist vector since this inverse utilizes the Euclidean norms of both the joint-rate vector and the twist vector. But the twist is not a Euclidean space (and neither is the joint-rate vector when the manipulator is composed of both revolute and prismatic joints). This problem has been addressed in these above papers and extensively in [19] by using the (weighted) generalized-inverse along with metrics on both the twist (M_v) and joint rates (M_q).

From (1.68)-(1.70) and (1.79)-(1.80), the pseudo-inverse and generalized-inverse of the manipulator Jacobian [19] are

$$
\begin{aligned}
J^{\dagger} &\stackrel{?}{=} C^{\tau}(F^{\tau}JC^{\tau})^{-1}F^{\tau} && (4.1)\\
&\stackrel{?}{=} C^{\tau}(CC^{\tau})^{-1}(F^{\tau}F)^{-1}F^{\tau} && (4.2)\\
&\stackrel{?}{=} C^{\dagger}F^{\dagger} \ , && (4.3)
\end{aligned}
$$

and

$$
\begin{aligned}
J^{\#} &= M_q^{-1}C^{\tau}(F^{\tau}M_vJM_q^{-1}C^{\tau})^{-1}F^{\tau}M_v && (4.4)\\
&= \left[M_q^{-1}C^{\tau}(CM_q^{-1}C^{\tau})^{-1}\right]\left[(F^{\tau}M_vF)^{-1}F^{\tau}M_v\right] && (4.5)\\
&= C^{\#}F^{\#} \ , && (4.6)
\end{aligned}
$$

respectively. A full-rank factorization of J, $J = FC$, is used in the above equations, where $F \in \Re^{(6\times r)}$ has full column rank r , $C \in \Re^{(r\times n)}$ has full row rank r, and n is the number of joints in the manipulator.

Two special cases of the generalized-inverse of a Jacobian are obtained when J is either full row rank or full column rank, *i.e.,*

$$J^{\#} = M_q^{-1}J^{\tau}(JM_q^{-1}J^{\tau})^{-1} \quad , J \text{ full row rank} \tag{4.7}$$

$$J^{\#} = (J^{\tau}M_vJ)^{-1}J^{\tau}M_v \quad , J \text{ full column rank,} \tag{4.8}$$

where (4.7) is found by letting $F = I_6$ and (4.8) is found by letting $C = I_n$ in (4.5).

As stated earlier, the metrics must be positive definite, and for invariance to coordinate transformations and scaling, the metrics must transform according to (1.84), *i.e.,*

$$M_{v'} = G_v^{\tau}M_vG_v \ \text{ for } V' = G_vV, \tag{4.9}$$

$$M_{q'} = G_q^{\tau}M_qG_q \ \text{ for } \dot{q}' = G_q\dot{q}. \tag{4.10}$$

If the desired twist is in the range of the Jacobian, then no metric on the twists is necessary since the residual $V - J\dot{q}_s$ is zero, *i.e.,*

$$J^{\#} = \left[M_q^{-1}C^{\tau}(CM_q^{-1}C^{\tau})^{-1}\right]\left[(F^{\tau}F)^{-1}F^{\tau}\right] \quad , \quad V \in \text{Range}[J] \ . \tag{4.11}$$

This equation is found by substituting $M_v = I_6$ in (4.5).

If the Jacobian has full column rank, then no metric on joint rates is necessary and (4.8) may be used.

If the conditions of both (4.11) and (4.8) are valid—*i.e.,* V is in the range of J and J has full column rank—then neither metric is needed and the generalized-inverse is equal to the pseudo-inverse,

$$J^{\#} = J^{\dagger} \quad , V \in \text{Range}[J] \text{ and } J \text{ full column rank.} \tag{4.12}$$

But, since all manipulators (including redundant manipulators) have singular configurations [4], and at singular configurations there exist V's not in the range of J, every manipulator has configurations in which a twist metric is needed.

For redundant manipulators, where J has full row rank except in singular configurations, the generalized-inverse is independent of the twist metric and (4.7) may be used. Furthermore, if all joints are revolute (or all are prismatic) the metric on the joint space is not needed for physical consistency—and the pseudo-inverse can be used—but the metric is needed for invariance to coordinate transformations and scaling.

For noncommensurate manipulators with J full row rank, the pseudo-inverse will generally be physically inconsistent (and not invariant to coordinate transformations and scaling) since the minimum norm $|\dot{q}|$ is physically inconsistent.

Sections 4.1-4.2 will discuss the situations in which the pseudo-inverse solution is physically consistent, invariant to scaling, and invariant to rigid body transformations.

4.1 Physical Consistency of $J^\dagger$

Although the pseudo-inverse of the manipulator Jacobian may be physically consistent in a given frame, there may be other frames in which $J^\dagger$ is not physically consistent. (This was suggested by equations (3.9), (3.10), (3.11), all of which have the terms $J^\tau J$ or JJ^τ embedded in them, and Section 3.2 which discussed the possible physical inconsistencies of these matrices.)

4.1.1 Rotations and Consistency of $J^\dagger$

Theorem 7 shows that if the pseudo-inverse is physically consistent in a given frame then it will remain physically consistent under any rigid body rotation.

Theorem 7 If the pseudo-inverse of J in frame i (${}^iJ^\dagger$) is physically consistent, then for every rigid body rotation *from frame i to frame j the pseudo-inverse of J in frame j (${}^jJ^\dagger$) is physically consistent.*

<u>Proof</u>

Let ${}^iV$ and ${}^jV$ be twists such that frame j is a rotation of frame i (no translation), ${}^jV = {}^jG_i\,{}^iV$.

Assume that the pseudo-inverse of ${}^iJ$ is physically consistent. The pseudo-inverse of the Jacobian in frame i is

$$ {}^iJ^\dagger = C^\tau(CC^\tau)^{-1}(F^\tau F)^{-1}F^\tau \quad , \tag{4.13} $$

where $J = FC$ is a full-rank factorization, F full column rank and C full row rank. The pseudo-inverse of the Jacobian in frame j is

$$ \begin{aligned} {}^jJ^\dagger &= ({}^jG_i{}^iJ)^\dagger = \left[({}^jG_iF)C\right]^\dagger & (4.14) \\ &= C^\tau(CC^\tau)^{-1}(F^\tau\,{}^jG_i^\tau\,{}^jG_i\,F)^{-1}F^\tau\,{}^jG_i^\tau & (4.15) \\ &= C^\tau(CC^\tau)^{-1}(F^\tau F)^{-1}F^{\tau i}G_j & (4.16) \\ &= {}^iJ^\dagger\,{}^iG_j \quad , & (4.17) \end{aligned} $$

where (4.16) follows from (4.15) since ${}^jG_i^\tau = ({}^jG_i)^{-1} = {}^iG_j$ for the case under discussion of ${}^jG_i$ a rotation (with no translation). It is now only necessary to prove that ${}^iJ^\dagger\,{}^iG_j$ is physically consistent.

Partition the pseudo-inverses in frames i and j into two $n \times 3$ matrices, W and X, and Y and Z, respectively,

$$ \begin{aligned} {}^iJ^\dagger &= [W\ X] & (4.18) \\ {}^jJ^\dagger &= [Y\ Z] = [WR\ \ XR] \quad , & (4.19) \end{aligned} $$

where $R = {}^iR_j$. Since ${}^iJ^\dagger$ operates on ${}^iV = [v^\tau, \omega^\tau]^\tau$, then each component in a row of W (or a row of X) must have like units or have zero value. Since R is dimensionless, the units of the elements in a row of Y (or Z) are identical to the units of the elements

Table 4.1. D-H parameters for PR virtual manipulator.

Joint Type	d	a	θ	α
P	d_1	0	0	0
R	d_2	0	θ_2	0

in a row of W (or X) and are therefore of consistent physical dimension. Therefore ${}^{j}J^{\dagger}$ is physically consistent.

■

Decouple frames are therefore actually decouple points, points at which the pseudo-inverse of the manipulator Jacobian (with respect to any frame at the decouple point) is physically consistent. The reason this point is called a decouple point will be made clear in Chapter 6.

Definition 2 A point is called a *decouple point* of a manipulator if the pseudo-inverse of the manipulator Jacobian in any frame located at this point is physically consistent.

4.1.2 Translations and Consistency of $J^{\dagger}$

A rigid body translation may cause a physically consistent $J^{\dagger}$ to become physically inconsistent. An example will demonstrate this fact.

The virtual manipulator [25] associated with the peg-in-the-hole problem [19, 37] after insertion has begun is shown in Figure 4.1. This PR manipulator has the Denavit-Hartenberg parameters given in Table 4.1.

The Jacobian in frame 2 is

$$ {}^{2}J = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{\tau} \tag{4.20} $$

and the pseudo-inverse in this frame, ${}^{2}J^{\dagger} = {}^{2}J^{\tau}$, is physically consistent.

In an arbitrarily translated frame (no rotation) the Jacobian is ${}^{t}J = ({}^{2}G^{t,2})\ {}^{2}J$, where

$$ {}^{2}G^{t,2} = \begin{bmatrix} I_3 & [p\times] \\ [0]_{3,3} & I_3 \end{bmatrix} \tag{4.21} $$

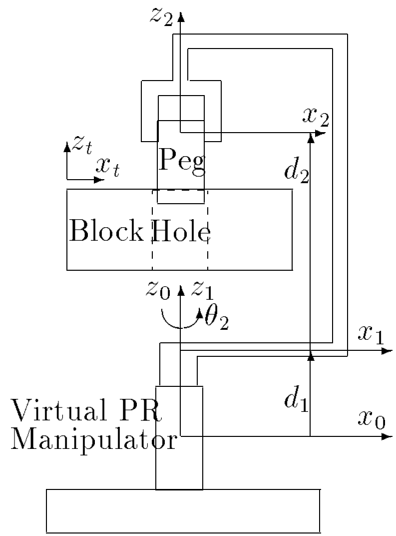

Figure 4.1. Peg-in-the-hole with PR virtual manipulator.

and $p = [p_x,\ p_y\ , p_z]^\tau$. The Jacobian in this arbitrarily translated frame is

$$^tJ = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ p_y & -p_x & 0 & 0 & 0 & 1 \end{bmatrix}^\tau \ , \tag{4.22}$$

and the pseudo-inverse is

$$^tJ^\dagger \stackrel{?}{=} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{p_y}{1+p_x^2+p_y^2} & \frac{-p_x}{1+p_x^2+p_y^2} & 0 & 0 & 0 & \frac{1}{1+p_x^2+p_y^2} \end{bmatrix} \ . \tag{4.23}$$

Note the physical inconsistency in the denominator of the terms in $^tJ^\dagger$. The physical inconsistency of this virtual manipulator model of the peg-in-the-hole problem is an alternative demonstration for the non-validity of the Mason-Raibert hybrid control techniques stated in published research [19, 22, 24].

4.1.3 Consistency of $J^\dagger$ in All Frames

The SCARA manipulator (Selective Compliant Articulated Robot for Assembly) [11] in Figure 4.2, with Denavit-Hartenberg parameters in Table 4.2 has a frame 2

Table 4.2. D-H parameters for the SCARA manipulator.

Joint Type	d	a	θ	α
R	0	a_1	θ_1	0
R	0	a_2	θ_2	0
R	0	0	θ_3	0
P	d_4	0	0	0

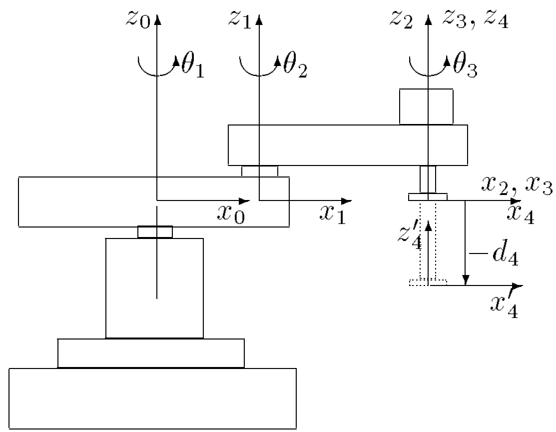

Figure 4.2. SCARA manipulator.

Jacobian of

$$
{}^2J = \begin{bmatrix} a_1 s_2 & 0 & 0 & 0 \\ a_2 + a_1 c_2 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} . \tag{4.24}
$$

Translating the frame of expression of the manipulator by an arbitrary vector p, results in a Jacobian, ${}^tJ = ({}^2G^{t,2})\,{}^2J$, whose pseudo-inverse is

$$
{}^tJ^{\dagger} = \begin{bmatrix} \frac{1}{a_1 s_2} & 0 & 0 & 0 & 0 & \frac{-p_y}{a_1 s_2} \\ -\frac{a_2 + a_1 c_2}{a_1 a_2 s_2} & \frac{1}{a_2} & 0 & 0 & 0 & \frac{a_2 p_y + a_1 c_2 p_y + a_1 p_x s_2}{a_1 a_2 s_2} \\ \frac{c_2}{a_2 s_2} & \frac{-1}{a_2} & 0 & 0 & 0 & \frac{-c_2 p_y + a_2 s_2 - s_2 p_x}{a_2 s_2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} . \tag{4.25}
$$

Since this pseudo-inverse is physically consistent, the pseudo-inverse in any translated or rotated frame (see Theorem 7) will be physically consistent for the SCARA manipulator.

The planar RRR manipulator, with its three revolute joints identical to the first three joints of the SCARA, also has a physically consistent pseudo-inverse in any frame. These two manipulators are often used as example manipulators to demonstrate new algorithms [2, 60]. Perhaps this is not appropriate, given their aforementioned special properties.

4.2 Invariance of $J^\dagger$ to Scaling

When the pseudo-inverse of the manipulator Jacobian is physically inconsistent, terms of unlike physical units are summed. If the parameters in this manipulator were re-scaled, perhaps from British to SI units, the physically inconsistent terms will cause the resulting pseudo-inverse to give a different result.

It has been argued that the problem of physical inconsistencies can be "factored out" by scaling the problem. The fallacy of this statement will presently be shown.

A *change of units scaling matrix* is a diagonal matrix that converts a physically consistent vector with physical units into a vector with similar physical units or no units. For example, if $V = [v^\tau, \omega^\tau]^\tau$, units$[v_x]$ = units$[v_y]$ = units$[v_z]$ = m/s, and units$[\omega_x]$ = units$[\omega_y]$ = units$[\omega_z]$ = rad/s, then S_v is a change of units scaling matrix if

$$S_v = \begin{bmatrix} \alpha_v I_3 & 0 \\ 0 & \alpha_\omega I_3 \end{bmatrix} , \qquad (4.26)$$

where, for example, α_v = (100cm/m)(60s/min) and α_ω = (60s/ min). The scaled twist, $V' = [\alpha_v v^\tau, \alpha_\omega \omega^\tau]^\tau$, has similar units to V, *i.e.,* each element of v and $\alpha_v v$ has units of L/T and each element of ω and $\alpha_\omega \omega$ has units of 1/T.

A manipulator joint-rate vector $\dot{q}$ should have the change of units joint-rate scaling matrix

$$S_q = \text{Diag}[e_1,\ e_2,\ \ldots,\ e_n] \ \text{, where } e_i = \begin{cases} \alpha_\omega, & \text{if joint } i \text{ is revolute} \\ \alpha_v, & \text{if joint } i \text{ is prismatic} \end{cases} , \qquad (4.27)$$

where the scalar physical unit transformations α_v and α_ω are the identical to those used in (4.26).

Any scaling of a physical unit for a single element of a noncommensurate vector must be identically scaled in all other elements of the noncommensurate vector. For instance, in the example discussed above, the time units were necessarily converted from seconds to minutes in both α_v and α_ω.

The change of units scaling matrix S_v is also normalizing if only the units—not the numerical value—of the noncommensurate vector is changed, *i.e.,* for the twist example above $\alpha_v = (\mathrm{s/m})$ and $\alpha_\omega = \mathrm{s}$. A normalizing units scaling matrix is numerically equal to the identity matrix, *e.g.,* $S_v \overset{N}{=} I_6$.

Scaling will now be applied to the inverse velocity problem. The twist vectors are scaled with the change of units diagonal scaling matrix S_v and the joint-rate vectors are scaled with the change of units diagonal scaling matrix S_q [15] such that

$$V_s = S_v V \tag{4.28}$$

$$\dot{q}_s = S_q \dot{q} \ . \tag{4.29}$$

The scaled version of the mapping of joint rates to twist of (1.1) is

$$V_s = S_v V = S_v J S_q^{-1} S_q \dot{q} = J_s \dot{q}_s \ , \tag{4.30}$$

where the scaled Jacobian is

$$J_s = S_v J S_q^{-1} \ . \tag{4.31}$$

To obtain the pseudo-inverse of J_s, first get the full rank factorization $J = FC$ so that $J_s = F_s C_s = (S_v F)(C S_q^{-1})$. Equation (4.2) is then used replacing all F's with F_s's and all C's with C_s's so that

$$(J_s)^\dagger \overset{?}{=} S_q^{-1} C^\tau (C S_q^{-2} C^\tau)^{-1} (F^\tau S_v^2 F)^{-1} F^\tau S_v \ . \tag{4.32}$$

The scaled joint-rate solution is thus

$$\dot{q}_{rs} \overset{?}{=} (J_s)^\dagger V_s \tag{4.33}$$

$$\overset{?}{=} (J_s)^\dagger S_v V \tag{4.34}$$

$$\overset{?}{=} S_q^{-1} C^\tau (C S_q^{-2} C^\tau)^{-1} (F^\tau S_v^2 F)^{-1} F^\tau S_v^2 V \ , \tag{4.35}$$

and the unscaled joint-rate solution is

$$\dot{q}_s' = S_q^{-1} q_{rs} \stackrel{?}{=} S_q^{-1}(J_s)^\dagger S_v V \tag{4.36}$$

$$\stackrel{?}{=} S_q^{-2} C^\tau (C S_q^{-2} C^\tau)^{-1} (F^\tau S_v^2 F)^{-1} F^\tau S_v^2 V \ . \tag{4.37}$$

Compare (4.37) with the generalized-inverse solution of $\dot{q}_s = J^\# V$ obtained using (4.5), *i.e.*,

$$\dot{q}_s = M_q^{-1} C^\tau (C M_q^{-1} C^\tau)^{-1} (F^\tau M_v F)^{-1} F^\tau M_v V \ . \tag{4.38}$$

It is evident that the two scaling matrices act as metrics where S_v^2 and S_q^2 in (4.37) correspond to the metrics M_v and M_q in (4.38), respectively. Since S_v^2 and S_q^2 are both positive definite and symmetric, they need only meet the additional requirements that $V \odot S_v^2 V$ and $\dot{q} \odot S_q^2 \dot{q}$ are physically consistent in order for the $\stackrel{?}{=}$ symbol in (4.37) to become an equal sign.

When the desired twist V is in the range of J, the solution $\dot{q}_s = J^\dagger V$ is always physically consistent. If $J^\dagger$ is physically inconsistent, the inconsistencies are canceled out when $J^\dagger$ is multiplied by V.

The RRRP-2 manipulator has a physically consistent pseudo-inverse in frame 0 and physically inconsistent pseudo-inverse in frame 2,

$${}^0J^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{c_1 c_{2+3}}{a_2 s_3} & \frac{s_1 c_{2+3}}{a_2 s_3} & \frac{s_{2+3}}{a_2 s_3} & \frac{s_1(a_1 s_{2+3} + a_2 s_3)}{a_2 s_3} & \frac{-c_1(a_1 s_{2+3} + a_2 s_3)}{a_2 s_3} & 0 \\ \frac{-c_1 c_{2+3}}{a_2 s_3} & \frac{-s_1 c_{2+3}}{a_2 s_3} & \frac{-s_{2+3}}{a_2 s_3} & \frac{-a_1 s_1 s_{2+3}}{a_2 s_3} & \frac{a_1 c_1 s_{2+3}}{a_2 s_3} & 0 \\ \frac{c_1 c_2}{s_3} & \frac{s_1 c_2}{s_3} & \frac{s_2}{s_3} & \frac{a_1 s_1 s_2}{s_3} & \frac{-a_1 c_1 s_2}{s_3} & 0 \end{bmatrix} , \tag{4.39}$$

$${}^2J^\dagger \stackrel{?}{=} \begin{bmatrix} 0 & 0 & \frac{-(a_1 + a_2 c_2)}{\beta} & \frac{s_2}{\beta} & \frac{c_2}{\beta} & 0 \\ \frac{c_3}{a_2 s_3} & \frac{1}{a_2} & 0 & 0 & 0 & 0 \\ \frac{-c_3}{a_2 s_3} & \frac{-1}{a_2} & 0 & 0 & 0 & 1 \\ \frac{1}{s_3} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} , \tag{4.40}$$

where $\beta = 1 + a_1^2 + a_2^2 c_2^2 + 2a_1 a_2 c_2$ is physically inconsistent. When the desired twist is in the range of J, the solution in each of the frames are identical and physically consistent. For instance, the twist for an arbitrary joint-rate vector, $\dot{q} = [\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3,\ \dot{q}_4]^\tau$, in each of frames 0 and 2 are

$$
{}^0V = \begin{bmatrix} c_1(a_2\dot{q}_3 s_2 + \dot{q}_4 s_{2+3}) \\ s_1(a_2\dot{q}_3 s_2 + \dot{q}_4 s_{2+3}) \\ -a_1(\dot{q}_2 + \dot{q}_3) - a_2 c_2 \dot{q}_3 - c_{2+3}\dot{q}_4 \\ s_1(\dot{q}_2 + \dot{q}_3) \\ -c_1(\dot{q}_2 + \dot{q}_3) \\ \dot{q}_1 \end{bmatrix} , \quad {}^2V = \begin{bmatrix} \dot{q}_4 s_3 \\ a_2\dot{q}_2 - c_3\dot{q}_4 \\ -\dot{q}_1(a_1 + a_2 c_2) \\ s_2\dot{q}_1 \\ c_2\dot{q}_1 \\ \dot{q}_2 + \dot{q}_3 \end{bmatrix} , \tag{4.41}
$$

where ${}^2V = {}^2G_0{}^0V$. Substituting ${}^0V$ and (4.39) into $\dot{q}_s = J^\dagger V$, and substituting ${}^2V$ and (4.40) into $\dot{q}_s = J^\dagger V$, both the solutions are $\dot{q}_s = \dot{q} = [\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3,\ \dot{q}_4]^\tau$. In frame 2, the physically inconsistent terms in ${}^2J^\dagger$ cancel when multiplied by any $V \in \mathrm{Range}[J]$.

For any twist not in the range of J, the solution is frame dependent. In frame 0 the solution is independent of scaling; in frame 2 the solution is not independent of scaling. For example, let the configuration be defined by

$$
\theta_1 = 0.1\mathrm{rad} \ , \quad \theta_2 = 0.2\mathrm{rad} \ , \quad \theta_3 = 0.3\mathrm{rad} \ , \quad d_4 = 4\mathrm{m} \tag{4.42}
$$

and let

$$
a_1 = 0.3\mathrm{m} \ , \quad a_2 = 1\mathrm{m} \ . \tag{4.43}
$$

Now consider the equivalent desired twists

$$
{}^0V_d = \begin{bmatrix} 2.4\frac{\mathrm{m}}{\mathrm{s}} \\ 0.2\frac{\mathrm{m}}{\mathrm{s}} \\ -7\frac{\mathrm{m}}{\mathrm{s}} \\ 0.6\frac{\mathrm{rad}}{\mathrm{s}} \\ -6\frac{\mathrm{rad}}{\mathrm{s}} \\ 1\frac{\mathrm{rad}}{\mathrm{s}} \end{bmatrix} , \quad {}^2V_d = {}^2G_0{}^0V = \begin{bmatrix} 1.329\frac{\mathrm{m}}{\mathrm{s}} \\ 0.4640\frac{\mathrm{m}}{\mathrm{s}} \\ -1.240\frac{\mathrm{m}}{\mathrm{s}} \\ 0.1967\frac{\mathrm{rad}}{\mathrm{s}} \\ 0.9805\frac{\mathrm{rad}}{\mathrm{s}} \\ 6.030\frac{\mathrm{rad}}{\mathrm{s}} \end{bmatrix} , \tag{4.44}
$$

not in the range of J. The solution for ${}^0V_d$ is

$$
\dot{q}_{sa} = {}^0J^\dagger\, {}^0V_d = \left[1.000\frac{\mathrm{rad}}{\mathrm{s}},\ 4.759\frac{\mathrm{rad}}{\mathrm{s}},\ 1.271\frac{\mathrm{rad}}{\mathrm{s}},\ 4.496\frac{\mathrm{m}}{\mathrm{s}}\right]^\tau \ . \tag{4.45}
$$

The resulting actual twist obtained by substituting this joint-rate vector into ${}^0V_{sa} = {}^0J\dot{q}_{sa}$ is

$$
{}^0V_{sa} = \left[2.396\frac{\mathrm{m}}{\mathrm{s}},\ 0.2404\frac{\mathrm{m}}{\mathrm{s}},\ -7.000\frac{\mathrm{rad}}{\mathrm{s}},\ 0.6020\frac{\mathrm{m}}{\mathrm{s}},\ -6.000\frac{\mathrm{rad}}{\mathrm{s}},\ 1.000\frac{\mathrm{rad}}{\mathrm{s}}\right]^\tau , \tag{4.46}
$$

which in frame 2 coordinates is ${}^2V_{sa} = {}^2G_0{}^0V_{sa}$,

$$ {}^2V_{sa} = \left[1.329\frac{\text{m}}{\text{s}},\ 0.4640\frac{\text{m}}{\text{s}},\ -1.280\frac{\text{m}}{\text{s}},\ 0.1987\frac{\text{rad}}{\text{s}},\ 0.9801\frac{\text{rad}}{\text{s}},\ 6.030\frac{\text{rad}}{\text{s}}\right]^{\tau} . \quad (4.47) $$

The solutions found in frame 0 will now be compared with those found in frame 2. The solution for ${}^2V_d$ is

$$ \dot{q}_{sb} \stackrel{?}{=} {}^2J^{\dagger 2}V_d \stackrel{?}{=} \begin{bmatrix} \frac{0.9686(0.6301\text{m}^2+1\text{m}^4)}{0.6103\text{m}^2\text{s}+1\text{m}^4\text{s}} \\ 4.759/\text{s} \\ 1.271/\text{s} \\ 4.496\text{m/s} \end{bmatrix} \text{, in frame 2,} \quad (4.48) $$

$$ \dot{q}_{sb} \stackrel{N}{=} [0.9805,\ 4.759,\ 1.271,\ 4.496]^{\tau} \text{ , using units of m and s.} \quad (4.49) $$

The joint-rate solution $\dot{q}_{sb}$ in (4.48) is physically inconsistent. The resulting actual twist obtained by using $\dot{q}_{sb}$ in ${}^2V_{sb} = {}^2J\dot{q}_{sb}$ is

$$ {}^2V_{sb} \stackrel{N}{=} [1.329,\ 0.4641,\ -1.255,\ 0.1948,\ 0.9610,\ 6.030]^{\tau} \ , \quad (4.50) $$

which transformed to frame 0 is

$$ {}^0V_{sb} = {}^0G_2{}^2V_{sb} \stackrel{N}{=} [2.396, 0.2404, -7.000, 0.6020, -6.000, 0.9805]^{\tau} \ . \quad (4.51) $$

These twists are different from the desired twists in (4.44).

If the twists are scaled according to (4.26) and the joint rates are scaled according to (4.27), where $\alpha_v = 100cm/m$ and $\alpha_\omega = 1$, then the numerical solution in frame 2 equals

$$ \dot{q}_{sc} \stackrel{N}{=} [0.9686,\ 4.759,\ 1.271,\ 449.6]^{\tau} \text{ , using units of cm and s.} \quad (4.52) $$

The resulting actual twist obtained by using $\dot{q}_{sc}$ in ${}^2V_{sc} = {}^2J\dot{q}_{sc}$ is

$$ {}^2V_{sc} \stackrel{N}{=} [132.8,\ 46.48,\ -124.0,\ 0.1924,\ 0.9493,\ 6.030]^{\tau} \ , \quad (4.53) $$

which transformed to frame 0 is

$$ {}^0V_{sc} = {}^0G_2{}^2V_{sc} \stackrel{N}{=} [239.6,\ 24.04,\ -700.0,\ 0.6020,\ -6.000,\ 0.9686]^{\tau} \ . \quad (4.54) $$

Notice that the results of (4.49) and (4.52) differ, *i.e.,* $\dot{q}_{sb} \neq \dot{q}_{sc}$. The first joint-rate components differ by more than 10%, the second and third joint rates are numerically identical, and the fourth joint-rate component (corresponding to the prismatic joint) in (4.52) is (as expected) 100 times the fourth component in (4.49). Since only terms in the first row of ${}^2J^\dagger$ in (4.40) are physically inconsistent, then only the first component of the joint-rate solution is adversely affected by scaling; the other components are scaled appropriately.

The solutions $\dot{q}_{sb}$ and $\dot{q}_{sc}$ are as "near" as they are only because the specified twist vector is "nearly" in the range of J, *i.e.,* the desired twist of (4.44) is "almost the same" (whatever that means!) as

$$ {}^0V = \left[2.501\frac{\mathrm{m}}{\mathrm{s}},\ 0.2510\frac{\mathrm{m}}{\mathrm{s}},\ -7.951\frac{\mathrm{m}}{\mathrm{s}},\ 0.4992\frac{\mathrm{rad}}{\mathrm{s}},\ -4.975\frac{\mathrm{rad}}{\mathrm{s}},\ 1.000\frac{\mathrm{rad}}{\mathrm{s}} \right]^\tau \quad (4.55) $$

$$ {}^2V = \left[1.182\frac{\mathrm{m}}{\mathrm{s}},\ -1.821\frac{\mathrm{m}}{\mathrm{s}},\ -1.280\frac{\mathrm{m}}{\mathrm{s}},\ 0.1987\frac{\mathrm{rad}}{\mathrm{s}},\ 0.9801\frac{\mathrm{rad}}{\mathrm{s}},\ 5.000\frac{\mathrm{rad}}{\mathrm{s}} \right]^\tau , \quad (4.56) $$

which are in the range of J.

The resulting actual twists V_{sb} and V_{sc} are not equal, are both different from the desired twist V_d, and are both also different form the physically consistent result found in V_{sa}.

For the special cases of unitless J, $J^\dagger$ is physically consistent.

Theorem 8 If J in some frame is unitless, then $J^\dagger$ in this frame is physically consistent.

Proof

Since the pseudo-inverse does not introduce any units not already in J, then $J^\dagger$ can have only the units of J and the inverse of the units of J or any combination of the two. Therefore, if J is unitless, then $J^\dagger$ is unitless.

■

For example, the Jacobians expressed in frames 1 and 2 for the SAR (PRP) manipulator,

$$
{}^1J = \begin{bmatrix} 0 & 0 & s_2 \\ 0 & 0 & -c_2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} , \qquad {}^2J = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{4.57}
$$

are unitless and the pseudo-inverses, ${}^1J^\dagger = {}^1J^\tau$ and ${}^2J^\dagger = {}^2J^\tau$, are physically consistent.

Of course the inverse of Theorem 8—*i.e.,* if J in some frame is not unitless, then $J^\dagger$ in this frame is not physically consistent—is not true. For example, the RRRP-2 manipulator has a physically consistent inverse in frame 0, yet the frame 0 Jacobian is not unitless.

Assume that the $\dot{q}_r = J^\dagger V$ is scaleable. Then rewriting (4.36), the scaled inverse velocity equation,

$$
\dot{q}'_s \stackrel{?}{=} (S_q^{-1}(J_s)^\dagger S_v)\ V \ , \tag{4.58}
$$

it is apparent that $(S_q^{-1}(J_s)^\dagger S_v)$ acts like $J^\dagger$ in the unscaled equation $\dot{q}_r \stackrel{?}{=} J^\dagger V$. When $(J_s)^\dagger$ is physically consistent, the $\stackrel{?}{=}$ can be replaced by an $=$ since scaleability means that $\dot{q}_r = \dot{q}'_s$. In this case,

$$
J^\dagger = S_q^{-1}(J_s)^\dagger S_v \ \text{, when } (J_s)^\dagger \text{ physically consistent.} \tag{4.59}
$$

Theorem 9 below must be used to verify this equation.

Theorem 9 *If D and E are physically consistent invertable diagonal matrices, then A is physically consistent if and only if DAE is physically consistent.*

Proof

Let $B = DAE$, where d_{ii} and e_{jj} are the diagonal elements of the diagonal matrices D and E, respectively. Then $b_{ij} = d_{ii}a_{ij}e_{jj}$. Since there is no addition in the equation for b_{ij} and no d_{ii} or e_{jj} is zero, then b_{ij} is physically consistent if and only if a_{ij} is physically consistent. Therefore, $B = DAE$ is physically consistent if A is physically consistent. The other direction of the proof follows directly from the fact that D^{-1} and E^{-1} are diagonal matrices and $A = D^{-1}BE^{-1}$ has the same form as $B = DAE$.

■

Theorem 9 and (4.59) tell us that if $(J_s)^\dagger$ is physically consistent, then $J^\dagger$ is physically consistent. Conversely, solve (4.59) for $(J_s)^\dagger$,

$$S_q J^\dagger S_v^{-1} = (J_s)^\dagger \text{ , when } J^\dagger \text{ physically consistent,} \tag{4.60}$$

to show that if $J^\dagger$ is physically consistent, so is any scaling $(J_s)^\dagger$ of $J^\dagger$. These results lead us directly to the fact that

Fact 1 If $J^\dagger$ is physically consistent, the solution $\dot{q}_s = J^\dagger V$ is independent of scaling for all V.

If $J^\dagger$ is not physically consistent, then (4.60) is not valid, and the pseudo-inverse solution to the inverse velocity problem is not scaleable.

The results of this section can be summarized as follows. A real physical system is always scaleable, *e.g.,* $V = J\dot{q}$ can always be scaled. The inverse velocity solution, $\dot{q}_r \stackrel{?}{=} J^\dagger V$, is scaleable for all twists if and only if $J^\dagger$ is physically consistent; in this case $\dot{q}_r = J^\dagger V$. If $J^\dagger$ is physically consistent, *i.e.,* the frame of expression has its origin at a decouple point, then scaling will not affect the resulting joint rates and the solution $\dot{q}_r$ is independent of scaling.

4.3 Equivalent Generalized Inverses

If an identity metric is assumed in a particular frame, the pseudo-inverse is equal to the generalized inverse. But in addition, there are other metrics that also give the same result.

Table 4.3. D-H parameters for the PRP Small Assembly Robot (SAR).

Joint Type	d	a	θ	α
P	d_1	0	0	0
R	0	0	θ_2	$\pi/2$
P	d_3	0	0	0

Using Theorem 10 below, all metrics which result in identical joint velocities can be found. Theorem 10 stems from Theorem 2.2 in [19] and the facts that $JJ^{\#} = FF^{\#}$ and $J^{\#}J = C^{\#}C$. The proof of these Theorems is given in [19].

Theorem 10 *All statements in the left column are equivalent statements and all statements in the right column are equivalent statements [19]:*

$$JJ^{\#} = (JJ^{\#})^{\tau} \qquad J^{\#}J = (J^{\#}J)^{\tau} \tag{4.61}$$

$$M_v JJ^{\#} = JJ^{\#} M_v \qquad M_q J^{\#}J = J^{\#}JM_q \tag{4.62}$$

$$J^{\dagger} = J^{\#} \qquad J^{\dagger} = J^{\#} \tag{4.63}$$

$$M_v JJ^{\dagger} = JJ^{\dagger} M_v \qquad M_q J^{\dagger}J = J^{\dagger}JM_q \quad . \tag{4.64}$$

If we assume (4.63), that the pseudo-inverse is equal to the generalized inverse, then the left equation of (4.64) may be used to solve for all equivalent twist metrics,

$$M_v J\ J^{\dagger} - J\ J^{\dagger}\ M_v = 0 \quad . \tag{4.65}$$

For example, the PRP Small Assembly Robot (SAR) shown in Figure 4.3, with Denavit-Hartenberg parameters given in Table 4.3, has a pseudo-inverse in frame 2 of

$${}^2J^{\dagger} = {}^2J^{\tau} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad . \tag{4.66}$$

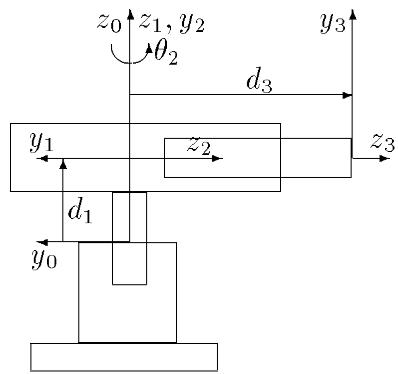

Figure 4.3. Small Assembly Robot (SAR).

Any metric of the form in (4.67) that is also positive definite will cause the generalized-inverse to equal the pseudo-inverse, *i.e.,*

$$M_v = \begin{bmatrix} m_{11} & 0 & 0 & m_{14} & 0 & m_{16} \\ 0 & m_{22} & m_{23} & 0 & m_{25} & 0 \\ 0 & m_{23} & m_{33} & 0 & m_{35} & 0 \\ m_{14} & 0 & 0 & m_{44} & 0 & m_{46} \\ 0 & m_{25} & m_{35} & 0 & m_{55} & 0 \\ m_{16} & 0 & 0 & m_{46} & 0 & m_{66} \end{bmatrix} . \tag{4.67}$$

The important result of this section is that if a pseudo-inverse is physically consistent, then there are a set of metrics which give identical results when using the generalized-inverse, *i.e.,* for every decouple point of a manipulator, a class of metrics exist for which the pseudo-inverse and generalized-inverse of the Jacobian are equal.

CHAPTER 5
MANIPULATOR MANIPULABILITY

As was discussed in Section 3.2.2, the matrices JJ^τ and $J^\tau J$ do not have physically consistent eigenvalues, eigenvectors, or a SVD. A few authors [17, 20, 31, 46, 60] have used other Jacobian functions—some Jacobian functions incorporating metrics—in manipulability definitions. In this section several of these manipulability ellipsoids will be introduced and their eigensystems will be explored.

There are three basic types of manipulability ellipsoids. Each of these arise from setting the square of the a Euclidean or non-Euclidean norm to less then or equal to 1. The manipulability ellipsoid discussed previously is called the *wrench manipulability ellipsoid* (or force manipulability ellipsoid) since this ellipsoid is defined as

$$|\tau|^2 = W^\tau(JJ^\tau)W \leq 1 \quad . \tag{5.1}$$

The "eigenvalues," λ_i, and "eigenvectors," e_i, of JJ^τ are used to create the ellipsoid with each principal axis in the direction of an e_i and axis length equal to $\sqrt{\frac{1}{\lambda_i}}$. A singular value decomposition of J can be used to deduce these same quantities (see Section 1.3).

As discussed previously in Section 3.2.2, this analysis is faulty due to the failure of JJ^τ to have a physically meaningful eigensystem (see Theorem 3).

It was proposed in [20] that incorporating a metric to replace the Euclidean norm of τ might correct this problem. The resulting equation if a metric is used to determine the M_τ-norm of τ_w is

$$|\tau_w|^2_{M_\tau} = W^\tau(JM_\tau J^\tau)W \leq 1 \quad . \tag{5.2}$$

It will now be shown that the ellipsoid defined by the eigenvalues and eigenvectors of $JM_\tau J$ does not meet the requirements for a physically consistent eigensystem.

The physical units of M_τ, found by forcing $\tau \odot M_\tau \tau$ to be physically consistent, are

$$\text{units}[M_\tau] = \frac{\gamma_\tau}{F^2 L^2} C \quad \text{, where } C = [c_{ij}] \text{ and} \tag{5.3}$$

$$c_{ij} = \begin{cases} U & \text{, joints } i \text{ and } j \text{ revolute} \\ L & \text{, either joint } i \text{ or } j \text{ revolute, the other prismatic} \\ L^2 & \text{, joints } i \text{ and } j \text{ prismatic.} \end{cases} \tag{5.4}$$

The units variable γ_τ is equal to the desired units of $|\tau|^2_{M_\tau}$.

With the above units for M_τ, the resulting units matrix for $J M_\tau J^\tau$ is

$$\text{units}[J M_\tau J^\tau] = \frac{\gamma_\tau}{F^2 L^2} \begin{bmatrix} [L^2]_{3,3} & [L]_{3,3} \\ [L]_{3,3} & [U]_{3,3} \end{bmatrix} . \tag{5.5}$$

The units matrix for $J M_\tau J^\tau$ is a scalar multiple of the units matrix of $J J^\tau$ for manipulators with all revolute joints—given in (3.26). Therefore, by Theorem 3, the wrench manipulability ellipsoid with metric M_τ is also based on a physically inconsistent eigensystem.

It should be pointed out that no metric is needed for a physically consistent $|\tau|$ if all the joints are of identical type, therefore the above result could have been immediately deduced.

A units analysis of the M_q metric used to make $|\dot{q}|^2_{M_q}$ physically consistent leads to the units matrix

$$\text{units}[M_q] = \gamma_q \frac{T^2}{L^2} C \quad \text{, where } C = [c_{ij}] \text{ and} \tag{5.6}$$

$$c_{ij} = \begin{cases} L^2 & \text{, joints } i \text{ and } j \text{ revolute} \\ L & \text{, joint } i \text{ or } j \text{ revolute, other prismatic} \\ U & \text{, joints } i \text{ and } j \text{ prismatic,} \end{cases} \tag{5.7}$$

where the units variable γ_q is equal to the desired units of $|\dot{q}|^2_{M_q}$. The units matrix M_q^{-1} is therefore

$$\text{units}[M_q^{-1}] = \frac{\gamma_q^{-1}}{T^2} C \quad \text{, where } C = [c_{ij}] \text{ and} \tag{5.8}$$

$$c_{ij} = \begin{cases} U & \text{, joints } i \text{ and } j \text{ revolute} \\ L & \text{, joint } i \text{ or } j \text{ revolute, other prismatic} \\ L^2 & \text{, joints } i \text{ and } j \text{ prismatic.} \end{cases} \tag{5.9}$$

This units matrix differs by a scalar constant from the units matrix of M_τ. Therefore, metrics derived for joint rates can be inverted and then used for joint torques, *i.e.*, $M^\tau = M_q^{-1}$.

The *twist manipulability ellipsoid* was defined originally [59] as

$$|\dot{q}_s|^2 = V^\tau \left((J^\dagger)^\tau J^\dagger\right) V \leq 1 \ . \tag{5.10}$$

The twist manipulability ellipsoid can alternatively be defined with a generalized-inverse and/or with a joint-rate metric as

$$|\dot{q}_s|^2_{M_q} = V^\tau \left((J^\#)^\tau J^\#\right) V \leq 1 \ , \tag{5.11}$$

$$|\dot{q}_s|^2_{M_q} = V^\tau \left((J^\dagger)^\tau M_q J^\dagger\right) V \leq 1 \ , \text{ or} \tag{5.12}$$

$$|\dot{q}_s|^2_{M_q} = V^\tau \left((J^\#)^\tau M_q J^\#\right) V \leq 1 \ . \tag{5.13}$$

Since noncommensurate manipulators generally have physically inconsistent $J^\dagger$ and thus can not have physically consistent eigensystems, only all revolute-jointed manipulators will be analyzed for the definitions in (5.10) and (5.12). The units analysis below for revolute joints using $J^\dagger$ and (5.12) is equivalent to the units analysis of any manipulator using $J^\#$ and (5.13).

Each of the n rows of $J^\dagger$ has the units

$$\text{units}[J^\dagger]_{(i,\cdot)} = [\frac{1}{L},\ \frac{1}{L},\ \frac{1}{L},\ U,\ U,\ U] \ \text{, for all revolute joints.} \tag{5.14}$$

(Notice that the rows of this $J^\dagger$ are ray coordinate screws as opposed to the axis coordinate screws of the columns of J.) Therefore, the units of $(J^\dagger)^\tau J^\dagger$ for an all revolute-jointed manipulator are

$$\text{units}[(J^\dagger)^\tau J^\dagger] = \frac{1}{L^2}\begin{bmatrix} [U]_{3,3} & [L]_{3,3} \\ [L]_{3,3} & [L^2]_{3,3} \end{bmatrix} \ \text{, for all revolute joints.} \tag{5.15}$$

And since for an all revolute joint manipulator the metric M_q is entirely composed of identical units, the units of $(J^\dagger)^\tau M_q J^\dagger$ are proportional to the units of $(J^\dagger)^\tau J^\dagger$. By Theorem 3, the matrix $(J^\dagger)^\tau J^\dagger$ does not have a physically meaningful eigensystem.

Replacing the pseudo-inverse of J with the weighted generalized inverse of J does not change the fact that the matrix $(J^{\#})^{\tau} M_q J^{\#}$ does not have a physically meaningful eigensystem. (The physical units of $J^{\#}$ are a scalar multiple of the units of $J^{\dagger}$ when $J^{\dagger}$ is physically consistent.) But the matrix $(J^{\#})^{\tau} J^{\#}$ is physically consistent even for noncommensurate manipulators.

The *dynamic-manipulability ellipsoid* [17, 20, 60] is derived from the manipulator dynamics equation

$$\tau = M(q)\ddot{q} + h(q, \dot{q}) + g(q) \quad , \tag{5.16}$$

where τ represents the generalized-force vector at the joints, $M(q)$ is a positive definite mass matrix, $\ddot{q}$ is the joint acceleration, $h(q, \dot{q})$ represents the Coriolis and centrifugal forces, and $g(q)$ represents the gravitational forces. Solving for $\ddot{q}$ results in

$$\ddot{q} = M^{-1} \left[\tau - h(q, \dot{q}) - g(q) \right] \quad , \tag{5.17}$$

where the dependency in $M(q)$ on q has been dropped for simplicity of notation.

The development here follows from [20] and is given here to demonstrate the method with which manipulability matrices have been derived. Differentiating $V = J\dot{q}$ with respect to time results in

$$\dot{V} = J\ddot{q} + \dot{J}\dot{q} \quad . \tag{5.18}$$

Again to simplify the notation, define A as the frame acceleration,

$$A = J\ddot{q} = \dot{V} - \dot{J}\dot{q} \quad , \tag{5.19}$$

and $\tilde{\tau}$ as

$$\tilde{\tau} = \tau - h(q, \dot{q}) - g(q) \quad . \tag{5.20}$$

Substituting (5.20) into (5.17) and the result into (5.19) yields

$$A = JM^{-1}\tilde{\tau} \quad . \tag{5.21}$$

Solving for $\tilde{\tau}$ we get

$$\tilde{\tau}_s \stackrel{?}{=} (JM^{-1})^{\dagger} A \ , \tag{5.22}$$

or

$$\tilde{\tau}_s = (JM^{-1})^{\#} A \ . \tag{5.23}$$

The M_τ-norm of $\tilde{\tau}_s$ (using only the generalized inverse since the pseudo-inverse may be physically inconsistent) is

$$\begin{aligned} |\tilde{\tau}_s|^2_{M_\tau} &= A^\tau \left([(JM^{-1})^{\#}]^\tau M_\tau (JM^{-1})^{\#}\right) A & (5.24) \\ &= \ddot{q}^\tau \left(J^\tau [(JM^{-1})^{\#}]^\tau M_\tau (JM^{-1})^{\# } J\right) \ddot{q} \ . & (5.25) \end{aligned}$$

If J has full column rank, then

$$\begin{aligned} (JM^{-1})^{\#} &= MJ^{\#} \ , \ J \text{ full column rank} & (5.26) \\ \tilde{\tau}_s &= MJ^{\#} A \ , \ J \text{ full column rank.} & (5.27) \end{aligned}$$

and

$$\begin{aligned} |\tilde{\tau}_s|^2_{M_\tau} &= (MJ^{\#}A)^\tau M_\tau (MJ^{\#}A) \ , \ J \text{ full column rank} & (5.28) \\ &= A^\tau \left((J^{\#})^\tau M^\tau M_\tau M J^{\#}\right) A \ , \ J \text{ full column rank} & (5.29) \end{aligned}$$

The dynamic-manipulability ellipsoid is found using (5.29) so that

$$|\tilde{\tau}_s|^2_{M_\tau} = A^\tau \left((J^{\#})^\tau M^\tau M_\tau M J^{\#}\right) A \leq 1 \ , \ J \text{ full column rank,} \tag{5.30}$$

and the ellipsoid is found from eigensystem of $(J^{\#})^\tau M^\tau M_\tau M J^{\#}$. As discussed previously, a metric M_q^{-1} can be used for M_τ. If $M_q = M$ so that $|\dot{q}|^2_{M_q}$ is the kinetic energy of the manipulator, then (5.30) reduces to

$$\begin{aligned} |\tilde{\tau}_s|^2_{M_\tau} &= A^\tau \left((J^{\#})^\tau M J^{\#}\right) A \leq 1 \ , \\ & \qquad J \text{ full column rank and } M_\tau = M^{-1}. & (5.31) \end{aligned}$$

The ellipsoid found from the eigensystem $J^{\#^\tau} M J^{\#}$ (J full column rank) is physically consistent but does not meet the criteria of a valid eigensystem in (2.15), since the units of this matrix are proportional to the units of (5.15). (Notice that the matrix defining the dynamic manipulability ellipsoid is identical to the matrix defining the twist manipulability ellipsoid.)

Let us look a little further $J^{\#^{\tau}} M J^{\#}$, the definition for the dynamic manipulability ellipsoid as originally developed in [59]. Expanding (5.31) by substituting (5.21) for A yields

$$|\tilde{\tau}_s|^2_{M_\tau} = \ddot{q}^\tau \left(J^\tau (J^{\#})^\tau M J^{\#} J \right) \ddot{q} \ \text{, } J \text{ full column rank, } M_\tau = M^{-1}. \tag{5.32}$$

But for full column rank J, $J^{\#} J = I_n$ and (5.32) to the trivial equation

$$|\tilde{\tau}_s|^2_{M_\tau} = \ddot{q}^\tau M \ddot{q} \ \text{, } J \text{ full column rank, } M_\tau = M^{-1}, \tag{5.33}$$

and the ellipsoid is dependent only on the metric. But since M has the units of M_q and M_q does not satisfy the conditions necessary for a valid eigensystem for noncommensurate manipulators, again the dynamic manipulability ellipsoid is shown to have an invalid eigensystem. Note that although M_q is unitless for commensurate manipulators and thus M_q has a valid eigensystem, the dynamic manipulability ellipsoid does not have a valid eigensystem even for commensurate manipulators.

For the case when J does not have full column rank, (5.24) is used to define the ellipsoid [60]. But again, a units analysis of the matrices shows that the eigensystem requirements are violated. This is also true for the expanded version of this ellipsoid determined by (5.25) when the manipulator is noncommensurate; but, if the manipulator is commensurate, each term of the matrix determining the ellipsoid has identical units and the eigensystem is physically meaningful.

To summarize, none of the manipulability ellipsoids possess geometric invariance. The wrench manipulability ellipsoid defined by the eigensystem of matrix $JM^\tau J^\tau$ is not valid for any manipulator. The twist manipulability ellipsoid originally defined by the eigensystem of $(J^\dagger)^\tau J^\dagger$ and subsequently modified to $(J^\dagger)^\tau M_q J^\dagger$ and then to $(J^{\#})^\tau M_q J^{\#}$, is not valid for any manipulators. The dynamic-manipulability ellipsoid, defined by the eigensystem of matrix $[(JM^{-1})^{\#}]^{\tau} M_{\tau} (JM^{-1})^{\#}$, is not a physically consistent eigensystem even for the case when J has full column rank. If J has full column rank and $M_{\tau} = M^{-1}$, this matrix product reduces to $(J^{\#})^{\tau} M_q J^{\#}$ which also does not have a valid eigensystem. An expansion of the dynamic-manipulability equation leads to $J^{\tau} \left((J^{\#})^{\tau} M_q J^{\#} \right) J = M_q$, which has a valid eigensystem if the manipulator is commensurate.

Although the existing manipulability theory has been shown to be invalid in all cases for manipulators with six or more joints, for manipulators with six or fewer joints, the scalar manipulability measure, $\mathrm{Det}[J^{\tau} J]$, is physically meaningful at decouple points. At decouple points, the manipulability measure is physically consistent (see equation (6.105)). Thus, when a decoupled coordinate frame is used, the manipulability of these manipulators in one configuration can be meaningfully compared to the manipulability at other configurations.

CHAPTER 6
DECOMPOSITION OF SPACES

Griffis recently introduced a special six dimensional spring for use as a wrist placed on a 6-jointed manipulator [26]. He thus created a wrench space via small displacements (or twists) creating a K-orthogonal complement to the twists of freedom, which he called the twists of compliance. With this technique Griffis and Duffy [28] showed that independent position and force control can be accomplished for a two-dimensional example and that the twists of compliance are in fact K-orthogonal complements to the twists of freedom. Without adding such a wrist, this chapter explores several techniques for twist and wrench space decomposition.

Let us assume that a twist space referenced to a particular coordinate system is decomposed into two manifolds, and one of these manifolds is the twists of freedom subspace, $\mathcal{V}_f = \text{Range}[J]$, as previously defined in (1.13). The other manifold is the *twists of nonfreedom*, $\mathcal{V}_{nf}$, introduced by Lipkin and Duffy [36] in their important article on the nature of twists and wrenches as screws.

The twists of nonfreedom are the twists that are not possible to accomplish in a given configuration. Lipkin and Duffy [36] define this as a "subspace which is the orthogonal complement of " the twists of freedom, although Duffy later repudiates this notion in [22]. But since $\mathcal{V}_f$ is a noncommensurate space, the orthogonal complement of $\mathcal{V}_f$ is not an appropriate manifold to introduce since it does not have the physical dimensions of a twist manifold. This manifold would have the strange property of dependence on the units of expression of $\mathcal{V}_f$. The wrenches of constraint subspace, $\mathcal{W}_c$, when viewed as a unitless vector space in $\Re^6$, is recognized as the orthogonal complement of an assumed unitless version of $\mathcal{V}_f$. But $\mathcal{W}_c$ only in special cases appear to have the physical units of twist vectors, which is necessary for the manifold $\mathcal{V}_{nf}$ to be meaningful. (To be fair, [36] defines twists of nonfreedom in the context of an example that appears to have a unitless basis for $\mathcal{W}_c$, which could therefore be viewed as an appropriate twist subspace. This dissertation defines wrenches of constraint in a manner consistent to the definition given in [36].)

6.1 Projections and Kinestatic Filters

In commensurate systems, the pseudo-inverse and generalized-inverse can be used to separate various spaces into two disjoint spaces [34, 56]. In noncommensurate systems, care must be taken when using the pseudo-inverse. If the pseudo-inverse is physically inconsistent, projections using this inverse are also generally physically inconsistent.

All types of projections for the various manipulator spaces are derived below using the generalized-inverse, although in cases of a physically consistent pseudo-inverse, the generalized-inverse may be replaced by the pseudo-inverse.

The twist space projection is found through the following series of equations:

$$V = J\dot{q} \tag{6.1}$$

$$\dot{q}_s = J^{\#}V_d \tag{6.2}$$

$$V_r = J\dot{q}_s \tag{6.3}$$

$$V_r = JJ^{\#}V_d \ , \tag{6.4}$$

where the s subscript is for "solution", the "d" subscript is for "desired', and the "r" subscript is for "resulting."

The joint-rate space projection, obtained by substituting (6.1) into (6.2), is

$$\dot{q}_s = J^{\#}J\dot{q}_d \ . \tag{6.5}$$

The wrench space projection is found through the following series of equations:

$$\tau = J^{\tau}W \tag{6.6}$$

$$W_s = J^{\#^\tau} \tau_d \tag{6.7}$$

$$W_s = J^{\#^\tau} J^\tau \tau_d = (JJ^\#)^\tau \tau_d \ . \tag{6.8}$$

The generalized-force space projection, obtained by substituting (6.7) into (6.6), is

$$\tau_r = J^\tau J^{\#^\tau} \tau_d = (J^\# J)^\tau \tau_d \ . \tag{6.9}$$

The various projection matrices are the four kinestatic filters [19],

$$P_v = JJ^\# \ , \quad P_q = J^\# J \ , \quad P_w = (JJ^\#)^\tau \ , \quad P_\tau = (J^\# J)^\tau \ . \tag{6.10}$$

The various spaces can now be decomposed into disjoint spaces using the above projection matrices and (6.4), (6.5), (6.8), and (6.9),

$$\mathcal{V} = \text{Null}[JJ^\#] \overset{M_v}{\oplus} \text{Range}[JJ^\#] \tag{6.11}$$

$$\mathcal{Q} = \text{Null}[J^\# J] \overset{M_q}{\oplus} \text{Range}[J^\# J] \tag{6.12}$$

$$\mathcal{W} = \text{Null}[(JJ^\#)^\tau] \overset{M_v^{-1}}{\oplus} \text{Range}[(JJ^\#)^\tau] \tag{6.13}$$

$$\mathcal{T} = \text{Null}[(J^\# J)^\tau] \overset{M_q^{-1}}{\oplus} \text{Range}[(J^\# J)^\tau] \ . \tag{6.14}$$

where the symbol $\overset{M_v}{\oplus}$ means that the two subspaces on either side of this symbol are M_v-orthogonal. The normal direct sum ($\oplus$) means that the two spaces are orthogonal (in the Euclidean sense). Notice that the above decompositions do not follow from the fundamental theorem of linear algebra, $\Re^m = \text{Null}[A^\tau] \oplus \text{Range}[A]$, where the range and null operators operate on a matrix and its transpose. For the metric-dependent decompositions, the range and null operators operate on the same matrix.

The above decomposition equations can be simplified by applying some facts about the full rank decomposition of the Jacobian, $J = FC$ and $J^\# = C^\# F^\#$ of (4.5),

$$JJ^\# = FCC^\# F^\# = FF^\# \tag{6.15}$$

$$J^\# J = C^\# F^\# FC = C^\# C \ , \tag{6.16}$$

and some facts about the null and range space operators,

$$\begin{aligned}
\mathrm{Null}[JJ^{\#}] &= \mathrm{Null}[FF^{\#}] = \mathrm{Null}[F^{\#}] && (6.17)\\
\mathrm{Range}[JJ^{\#}] &= \mathrm{Range}[FF^{\#}] = \mathrm{Range}[F] && (6.18)\\
\mathrm{Null}[J^{\#}J] &= \mathrm{Null}[C^{\#}C] = \mathrm{Null}[C] && (6.19)\\
\mathrm{Range}[J^{\#}J] &= \mathrm{Range}[C^{\#}C] = \mathrm{Range}[C^{\#}] \ . && (6.20)
\end{aligned}$$

Each of the statements in (6.17)-(6.20) can be proven in a manner similar to that shown below for (6.17).

Let $FF^{\#}x = 0$. Multiply both sides by $F^{\#}$ to give $F^{\#}FF^{\#}x = 0$. But by the property of the generalized-inverse given in (1.76), $F^{\#}FF^{\#} = F^{\#}$, so that $F^{\#}x = 0$. Therefore, $\mathrm{Null}[FF^{\#}] = \mathrm{Null}[F^{\#}]$.

These simplifications lead to the below simplified decomposition equations:

$$\begin{aligned}
V &= \mathrm{Null}[J^{\#}] \overset{M_v}{\oplus} \mathrm{Range}[J] && (6.21)\\
Q &= \mathrm{Null}[J] \overset{M_q}{\oplus} \mathrm{Range}[J^{\#}] && (6.22)\\
W &= \mathrm{Null}[J^{\tau}] \overset{M_v^{-1}}{\oplus} \mathrm{Range}[(J^{\#})^{\tau}] && (6.23)\\
T &= \mathrm{Null}[(J^{\#})^{\tau}] \overset{M_q^{-1}}{\oplus} \mathrm{Range}[J^{\tau}] \ , && (6.24)
\end{aligned}$$

and the even simpler decomposition equations:

$$\begin{aligned}
V &= \mathrm{Null}[F^{\#}] \overset{M_v}{\oplus} \mathrm{Range}[F] && (6.25)\\
Q &= \mathrm{Null}[C] \overset{M_q}{\oplus} \mathrm{Range}[C^{\#}] && (6.26)\\
W &= \mathrm{Null}[F^{\tau}] \overset{M_v^{-1}}{\oplus} \mathrm{Range}[(F^{\#})^{\tau}] && (6.27)\\
T &= \mathrm{Null}[(C^{\#})^{\tau}] \overset{M_q^{-1}}{\oplus} \mathrm{Range}[C^{\tau}] \ . && (6.28)
\end{aligned}$$

Each metric will give a different decomposition. If the metric has the required property (that it transforms via a congruence transformation, (1.84)), then the frame of expression has no bearing on the decomposition.

The below two facts allow us, in some cases, to apply the above metric dependent decompositions, which use the generalized-inverse, to a metric independent decomposition, which uses the pseudo-inverse.

Fact 2 If $J^\dagger = J^\#$ for some metric M_v and some metric M_q, then $J^\dagger$ is physically consistent.

Fact 3 If $J^\dagger$ is physically consistent, then $J^\dagger = J^\#$ for some metric M_v and some metric M_q.

If the pseudo-inverse is used instead of the generalized-inverse by choosing change of unit identity scaling metrics for M_v and M_q, the decomposition is frame dependent and only valid if the pseudo-inverse is physically consistent. The decomposition for physically consistent $J^\dagger$ is

$$\mathcal{V} = \mathrm{Null}[JJ^\dagger] \oplus \mathrm{Range}[JJ^\dagger] \tag{6.29}$$

$$\mathcal{Q} = \mathrm{Null}[J^\dagger J] \oplus \mathrm{Range}[J^\dagger J] \tag{6.30}$$

$$\mathcal{W} = \mathrm{Null}[(JJ^\dagger)^\tau] \oplus \mathrm{Range}[(JJ^\dagger)^\tau] \tag{6.31}$$

$$\mathcal{T} = \mathrm{Null}[(J^\dagger J)^\tau] \oplus \mathrm{Range}[(J^\dagger J)^\tau] \ . \tag{6.32}$$

From Theorem 10 and the fact that $J^\dagger = J^\#$ for some metric (since $J^\dagger$ is assumed physically consistent), $JJ^\dagger = (JJ^\dagger)^\tau$ and $J^\dagger J = (J^\dagger J)^\tau$. Therefore the above decompositions simplify to

$$\mathcal{V} = \mathcal{W} = \mathrm{Null}[JJ^\dagger] \oplus \mathrm{Range}[JJ^\dagger] \tag{6.33}$$

$$\mathcal{Q} = \mathcal{T} = \mathrm{Null}[J^\dagger J] \oplus \mathrm{Range}[J^\dagger J] \ , \tag{6.34}$$

when $J^\dagger$ is physically consistent. The spaces $\mathcal{V}$ and $\mathcal{W}$ are decomposed identically as are the spaces $\mathcal{Q}$ and $\mathcal{T}$.

The above decomposition can be further simplified by using the below equations:

$$JJ^\dagger = FF^\dagger \tag{6.35}$$

$$J^\dagger J = C^\dagger C \tag{6.36}$$

$$\mathrm{Null}[JJ^\dagger] = \mathrm{Null}[FF^\dagger] = \mathrm{Null}[F^\dagger] = \mathrm{Null}[(F^\tau F)^{-1}F^\tau] \tag{6.37}$$

$$= \mathrm{Null}[F^\tau] = \mathrm{Null}[C^\tau F^\tau] = \mathrm{Null}[J^\tau] \tag{6.38}$$

$$\mathrm{Range}[JJ^\dagger] = \mathrm{Range}[J] \tag{6.39}$$

$$\mathrm{Null}[J^\dagger J] = \mathrm{Null}[J] \tag{6.40}$$

$$\mathrm{Range}[J^\dagger J] = \mathrm{Range}[C^\dagger C] = \mathrm{Range}[C^\dagger] = \mathrm{Range}[C^\tau(CC^\tau)^{-1}] \tag{6.41}$$

$$= \mathrm{Range}[C^\tau] = \mathrm{Range}[C^\tau F^\tau] = \mathrm{Range}[J^\tau]\ . \tag{6.42}$$

The space decompositions for frames in which $J^\dagger$ is physically consistent are therefore

$$\mathcal{V} = \mathcal{W} = \mathrm{Null}[J^\tau] \oplus \mathrm{Range}[J] \tag{6.43}$$

$$\mathcal{Q} = \mathcal{T} = \mathrm{Null}[J] \oplus \mathrm{Range}[J^\tau]\ . \tag{6.44}$$

Equations (6.43) and (6.44) appear to be direct applications of the fundamental theorem of linear algebra; this is a deceptive notion. The reader should remember the limited scope of these equations—*i.e.,* they are only valid in frames in which $J^\dagger$ is physically consistent—and their rather involved derivations.

This decomposition will be explored further in the subsequent sections.

6.2 Twist Decomposition

In order to demonstrate the problem with defining a twist of nonfreedom manifold as a subspace, two examples will be shown. One example will show when these twists constitute a subspace and the other will show when they do not form a subspace.

First consider the SCARA manipulator of Figure 4.2. The SCARA Jacobian expressed in frame 2 coordinates was given in (4.24). The column-reduced echelon form of the wrench of constraint subspace in this frame, $\mathcal{W}_c = \text{Null}[{}^2J^\tau]$, is

$$\left[{}^2\mathcal{W}_c\right]_b E_w = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^\tau \quad , \tag{6.45}$$

where E_w is the matrix that converts $[\mathcal{W}_c]_b$ to column-reduced echelon form. Note that these wrenches might also be interpreted as twists of nonfreedom with no discrepancy with units,

$$\left[{}^2\mathcal{V}_{nf}\right]_b E_v = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^\tau \quad . \tag{6.46}$$

The SAR (PRP) manipulator of Figure 4.3 has the Jacobian and wrench of constraint subspace basis vectors expressed in frame 3 coordinates of

$${}^3J = \begin{bmatrix} 0 & d_3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad \left[{}^3\mathcal{W}_c\right]_b E_w = \begin{bmatrix} 0 & -\frac{1}{d_3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad . \tag{6.47}$$

Note that these basis wrenches cannot be interpreted as twists of nonfreedom since the second basis vector does not have the units of a twist (an axis coordinate screw). Therefore, for this manipulator expressed in frame 3 coordinates, the concept of twists of nonfreedom as described previously (as a subspace) is untenable.

A slightly modified definition of twists of nonfreedom is therefore necessary and is given below.

<u>*Definition 3*</u> *Twists of nonfreedom* are twists that the manipulator cannot fully generate in a given configuration,

$$\mathcal{V}_{nf} = \mathcal{V} - \mathcal{V}_f \quad . \tag{6.48}$$

The meaning of the above equation might need explanation. The manifold $\mathcal{V}_{nf}$ include all the twists of $\mathcal{V}$ except those twists in $\mathcal{V}_f$. This is not the orthogonal complement of $\mathcal{V}_f$, which (as stated previously) is physically inconsistent for screws.

The nonfreedom twist manifold might also be defined as

$$\mathcal{V}_{nf} = \{V_{nf} : \ V_{nf} \in \mathcal{V} \text{ and } V_{nf} \notin \mathcal{V}_f\} \ . \tag{6.49}$$

In general, the manifold $\mathcal{V}_{nf}$ is not a subspace. Typically, two twists of nonfreedom might sum to a twist of freedom or a nonfreedom twist.

For example, two nonfreedom twists for the SAR manipulator expressed in frame 3 coordinates are

$$ {}^3V_{nf}^a = \begin{bmatrix} 0, \\ 0 \\ 1\frac{\mathrm{m}}{\mathrm{s}} \\ 1\frac{\mathrm{rad}}{\mathrm{s}} \\ 0 \\ 0 \end{bmatrix} \ , \quad {}^3V_{nf}^b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1\frac{\mathrm{rad}}{\mathrm{s}} \\ 0 \\ 0 \end{bmatrix} \ . \tag{6.50}$$

The sum of these two nonfreedom twists is the twist of freedom $[0,\ 0,\ 1\frac{\mathrm{m}}{\mathrm{s}},\ 0,\ 0,\ 0]^\tau$. The difference of these two nonfreedom twists is the nonfreedom twist $[0,\ 0,\ 1\frac{\mathrm{m}}{\mathrm{s}},\ 2\frac{\mathrm{rad}}{\mathrm{s}},\ 0,\ 0]^\tau$.

Since $\mathcal{V}_{nf}$ is not, in general, a subspace, a direct sum decomposition of twists of freedom and twists of nonfreedom is not typically possible, *i.e.*,

$$\mathcal{V} \neq \mathcal{V}_f \oplus \mathcal{V}_{nf} \ . \tag{6.51}$$

In the special cases when $\mathcal{W}_c$ can be interpreted entirely as twists, the twist space can be decomposed as the direct sum decomposition, $\mathcal{V} = \mathcal{V}_f \oplus \mathcal{V}_i$, where $\mathcal{V}_i$ are the subspace of inaccessible twists defined below.

<u>*Definition 4*</u> *Inaccessible twists* constitute the screw subspace of twists such that

$$\mathcal{V}_i \subseteq \mathcal{V}_{nf} \ , \tag{6.52}$$

and the inner product $V_i \odot V_f \stackrel{?}{=} v_i \odot v_f + \omega_i \odot \omega_f$ (which is generally physically inconsistent) is physically consistent for any $V_i \in \mathcal{V}_i$ and any $V_f \in \mathcal{V}_f$. The subspace $\mathcal{V}_i$ may not exist.

If $\mathcal{V}_i = \mathcal{V}_{nf}$, then the twist space is uniquely decomposed by the direct sum decomposition

$$\mathcal{V} = \mathcal{V}_f \oplus \mathcal{V}_i \ \text{, if } \mathcal{V}_i = \mathcal{V}_{nf}. \tag{6.53}$$

6.3 Wrench Decomposition

Assume that a wrench space referenced to a particular coordinate system is decomposed into two manifolds. One of these manifolds equals the wrenches of constraint, $\mathcal{W}_c = \text{Null}[J^T]$, as previously defined in (1.19). Since $\mathcal{W}_c$ is the null space of a matrix, it must be a subspace. The other manifold is the *wrenches of nonconstraint* manifold [36], $\mathcal{W}_{nc}$.

The wrenches of nonconstraint, when applied at the end effector of a manipulator, require some nonzero joint forces for static balancing or will cause some motion of the manipulator. Lipkin and Duffy [36] define wrenches of nonconstraint in an analogous fashion to the twists of nonfreedom, *i.e.,* according to [36], the wrench of nonconstraint manifold is the orthogonal complement of the wrench of constraint subspace. But the orthogonal complement of $\mathcal{W}_c$ has physical dimensions of a twist manifold. Furthermore, when $\mathcal{W}_c$ is viewed as a unitless vector space in $\Re^6$, the orthogonal complement is a unitless version of $\mathcal{V}_f$. But the axis screw vectors of $\mathcal{V}_f$, only in special cases appear to have the physical units of wrench vectors, a necessary requirement for the manifold $\mathcal{W}_{nc}$ to be meaningful.

For example, the orthogonal complement of ${}^3\mathcal{W}_c$ for the SAR manipulator is the Jacobian, ${}^3J$, given in (6.47). The second basis vector of ${}^3J$ in (6.47) is obviously not a wrench (a ray coordinate screw), so this subspace cannot describe wrenches of nonconstraint.

To avoid the above problems, a slight modification of the definition of wrenches of nonconstraint is given below.

<u>*Definition 5*</u> *Nonconstraint wrenches* are wrenches that will produce a nonzero power with some twist of freedom [25],

$$\mathcal{W}_{nc} = \mathcal{W} - \mathcal{W}_c \ . \tag{6.54}$$

The manifold of nonconstraint wrenches are all the wrenches of $\mathcal{W}$ except those wrenches in $\mathcal{W}_c$. This is not, in general, the orthogonal complement of $\mathcal{W}_c$, which (as stated previously) is physically inconsistent for screws.

Wrenches of nonconstraint might also be defined as

$$\mathcal{W}_{nc} = \{W_{nc} : \ W_{nc} \in \mathcal{W} \text{ and } W_{nc} \notin \mathcal{W}_c\} \ . \tag{6.55}$$

Note that $\mathcal{W}_{nc}$ is a manifold that, in general, is not a subspace, so that no direct sum decomposition of wrenches of constraint and wrenches of nonconstraint is generally possible, *i.e.,*

$$\mathcal{W} \neq \mathcal{W}_c \oplus \mathcal{W}_{nc} \ . \tag{6.56}$$

In the special cases when the twists of Null$[W_c^T]$ possess a meaningful interpretation as $\mathcal{W}_{nc}$, the wrench space can be decomposed via the direct sum decomposition, $\mathcal{W} = \mathcal{W}_c \oplus \mathcal{W}_d$, where $\mathcal{W}_d$ are the subspace of driving wrenches defined below.

<u>*Definition 6*</u> *Driving wrenches* constitute the screw subspace of wrenches such that

$$\mathcal{W}_d \subseteq \mathcal{W}_{nc} \ , \tag{6.57}$$

and the inner product $W_d \odot W_{nc} \stackrel{?}{=} f_d \odot f_{nc} + n_d \odot n_{nc}$ (which is generally physically inconsistent) is physically consistent for any $W_d \in \mathcal{W}_d$ and any $W_{nc} \in \mathcal{W}_{nc}$. The subspace $\mathcal{W}_d$ may not exist.

If $\mathcal{W}_d = \mathcal{W}_{nc}$, then the wrench space is uniquely decomposed by the direct sum decomposition

$$\mathcal{W} = \mathcal{W}_d \oplus \mathcal{W}_c \ , \text{ if } \mathcal{W}_d = \mathcal{W}_{nc}. \tag{6.58}$$

If both twists of nonfreedom and wrenches of nonconstraint are subspaces (and thus are identically the inaccessible twists and the driving wrenches, respectively), then a hybrid control is accomplished by decomposing the desired twist into twists of freedom and twists of nonfreedom and the desired wrench into wrenches of constraint and wrenches of nonconstraint, and then filtering out the inaccessible twists and constraint wrenches. This assures that the control inputs will be entirely composed of twists of freedom and driving wrenches.

6.4 Hybrid Control

The hybrid control algorithms of Mason [40, 41] and Raibert [51] inherently assume a decomposition essentially equivalent to

$$\Re^6 \stackrel{?}{=} \mathrm{Range}[J] \oplus \mathrm{Null}[J^T] = \mathcal{V}_f \oplus \mathcal{W}_c \ . \tag{6.59}$$

This theory splits the hybrid control problem into "natural" and "artificial" constraints at what is now commonly know as the "center of compliance" or "compliance center" [2, 25, 60] (called a constraint frame in [11, 51]). A *center of compliance* is defined as a point through which pure forces produce only pure translations and pure couples produce only pure rotations about that point. This point may or may not exist, or may exist at more than one point.

When the coordinate reference frame origin is located at the center of compliance, the MRHCT (Mason and Raibert's hybrid control theory) states that the diagonal selection matrices [11, 51] are used to determine the appropriate action for each loop of the hybrid position and force control., *i.e.,* each joint is used to control either a position component (twist) or a force component (wrench).

The MRHCT calls these two subspaces orthogonal complements, which these subspaces appear to be if the screw spaces were instead commensurate six dimensional vector subspaces as in (1.61). But they are not orthogonal complement screw subspaces.

An example will now demonstrate the MRHCT [1, 2, 19]. The task at hand is to place a peg into a hole as shown in Figure 4.1. (In this example, the virtual PR manipulator of the figure is not involved.) The "natural" and "artificial" constraints, taken together (since the distinction between the two is sometimes open to interpretation), with respect to frame 2 are $v_x = v_y = 0$, $f_z = 0$, and $n_z = 0$. Both the twist and wrench selection matrices are diagonal matrices, both with elements of either 0 or 1. This leads to the twist selection matrix, ${}^2P_v$, and the wrench selection matrix ${}^2P_w$, *i.e.*,

$$
{}^2P_v = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad {}^2P_w = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad . \tag{6.60}
$$

The selection matrices are always related by the equation

$$
P_w = I_6 - P_v \quad . \tag{6.61}
$$

The hybrid control then filters the specified twist, V_s, and wrench, W_s, with the selection matrices as follows:

$$
{}^2V = {}^2P_v{}^2V_s \quad , \quad {}^2W = {}^2P_w{}^2W_s \quad . \tag{6.62}
$$

This guarantees that the twist ${}^2V \in {}^2\mathcal{V}_f$ and ${}^2W \in {}^2\mathcal{W}_c$ in frame 2.

It is apparent that the selection matrices, P_v and P_w, act as filters on twists and wrenches. In fact, P_v and P_w are projection matrices,

$$
{}^2P_v = {}^2B{}^2B^\dagger \;=\; {}^2B\left[{}^2B({}^2B^{\tau\,2}B)^{-1\,2}B^\tau\right] \tag{6.63}
$$

$$
{}^2P_w = {}^2C{}^2C^\dagger \;=\; {}^2C\left[{}^2C({}^2C^{\tau\,2}C)^{-1\,2}C^\tau\right] \quad , \tag{6.64}
$$

where B represents a basis for the twists of freedom and C represents a basis for the wrenches of constraint,

$$ {}^2B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad {}^2C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad . \tag{6.65} $$

In frame 2 the MRHCT seems to work. But in a frame t (see Figure 4.1), arbitrarily translated from frame 2, the MRHCT fails. In this frame the projection matrices, $P_v = {}^tJ{}^tJ^\dagger$ and $P_w = {}^tW_c{}^tW_c^\dagger$, are physically inconsistent, *i.e.*,

$$ {}^tP_v \stackrel{?}{=} \begin{bmatrix} \frac{p_y^2}{\gamma} & \frac{-p_xp_y}{\gamma} & 0 & 0 & 0 & \frac{p_y}{\gamma} \\ \frac{-p_xp_y}{\gamma} & \frac{p_x^2}{\gamma} & 0 & 0 & 0 & -\frac{p_x}{\gamma} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{p_y}{\gamma} & \frac{-p_x}{\gamma} & 0 & 0 & 0 & \frac{1}{\gamma} \end{bmatrix} \quad , \quad {}^tP_w \stackrel{?}{=} \begin{bmatrix} \frac{1+p_x^2}{\gamma} & \frac{p_xp_y}{\gamma} & 0 & 0 & 0 & \frac{-p_y}{\gamma} \\ \frac{p_xp_y}{\gamma} & \frac{1+p_y^2}{\gamma} & 0 & 0 & 0 & \frac{p_x}{\gamma} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{-p_y}{\gamma} & \frac{p_x}{\gamma} & 0 & 0 & 0 & \frac{p_x^2+p_y^2}{\gamma} \end{bmatrix} \quad , \tag{6.66} $$

where $\gamma \stackrel{?}{=} 1 + p_x^2 + p_y^2$, a physically inconsistent quantity.

6.5 Decomposition with Ray Coordinate Twist Space

Recently several authors [1, 24] have expanded a discussion on isotropic subspaces begun in [52] and greatly enhanced in [37]. These articles have attempted a different decomposition using four manifolds, two of which are the twists of freedom and wrenches of constraint. Manipulate the twists space via the Δ matrix so that the twists of freedom and wrenches of constraint subspaces are both defined using ray coordinate screws, *i.e.*,

$$ \mathcal{V}_f^{\text{ray}} = \Delta \mathcal{V}_f = \text{Range}(\Delta J) \quad . \tag{6.67} $$

The *radical manifold*, $\mathcal{R}$, is the screw manifold of the common elements in $\mathcal{V}_f^{\text{ray}}$ and $\mathcal{W}_c$,

$$ \mathcal{R} = \mathcal{V}_f^{\text{ray}} \cap \mathcal{W}_c \quad . \tag{6.68} $$

The *defect manifold*, $\mathcal{D}$, is the manifold not covered by $\mathcal{V}_f^{\text{ray}}$ and $\mathcal{W}_c$,

$$(\mathcal{V}_f^{\text{ray}} \cup \mathcal{W}_c) \cup \mathcal{D} = \$^6 \quad , \tag{6.69}$$

where $\$^6$ is the full 6-dimensional ray coordinate screw space.

Let us investigate how each of these manifolds relate to the others. As shown in Theorem 1, V_f and $\mathcal{W}_c$ are reciprocal subspaces. Since $\mathcal{V}_f^{\text{ray}}$ is the ray coordinate version of $\mathcal{V}_f$, then $\mathcal{V}_f^{\text{ray}}$ and $\mathcal{W}_c$ are also reciprocal subspaces. This theorem leads to the corollary below which states that the radical manifold is a self-reciprocal subspace. The proof for the theorem below is based in part on Theorem 1 which states that coordinate transformations do not affect the reciprocal product.

<u>*Corollary 4*</u> *The radical screw subspace $\mathcal{R}$ is self-reciprocal,*

$$r_i \circ r_j = 0 \quad , \quad \forall\, r_i, r_j \in \mathcal{R} \quad . \tag{6.70}$$

<u>Proof</u>

Since $r \in \mathcal{R}$, $r \in \mathcal{V}_f^{\text{ray}}$, and $r \in \mathcal{W}_c$, and all $V_f^{\text{ray}} \in \mathcal{V}_f^{\text{ray}}$ and $W_c \in \mathcal{W}_c$ are reciprocal ($V_f^{\text{ray}} \circ W_c = 0$) by Theorem 2, then $r_i \circ r_j = 0$ for all i and j.

■

Since the screw subspace $\mathcal{R}$ is self-reciprocal, the screws in this subspace are self-reciprocal and mutually reciprocal. The theorem below also shows that each column of a manipulator Jacobian is self-reciprocal.

<u>*Theorem 11*</u> *For revolute and/or prismatic jointed manipulators, each column of a manipulator Jacobian is self-reciprocal.*

Proof

If the i-th joint in a manipulator is revolute, the i-th column of the manipulator Jacobian in frame $i-1$ is $[0,0,0,0,0,1]^\tau$. If the i-th joint in a manipulator is prismatic, the i-th column of the manipulator Jacobian in frame $i-1$ is $[0,0,1,0,0,0]^\tau$. Since both these screws are self-reciprocal and reciprocity is invariant to coordinate transformations, then regardless of the frame, the i-th column of the Jacobian is self-reciprocal.

■

The radical is always a subspace since it is the intersection of two subspaces. But $\mathcal{V}_f^{\text{ray}} \cup \mathcal{W}_c$ is generally not a subspace as is shown in the below example.

The P50 manipulator with $\theta_2 = \theta_3 = \pi/2$ and $\theta_4 = 0$ has

$$[\mathcal{V}_f^{\text{ray}}]_b = \left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ a_3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ a_2 \\ a_3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ a_3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad [\mathcal{W}_c]_b = \left[\left\{ \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{matrix} \right\} \right]. \tag{6.71}$$

Summing the fifth screw of $\mathcal{V}_f^{\text{ray}}$ and γ times the only screw of $\mathcal{W}_c$ results in the vector $[0,\ -1,\ 0,\ \gamma,\ 0,\ 0]^\tau$, for all γ, where units$[\gamma]$ = L. This screw is not in $\mathcal{V}_f^{\text{ray}} \cup \mathcal{W}_c$ for any nonzero γ. Therefore $\mathcal{V}_f^{\text{ray}} \cup \mathcal{W}_c$ is not a screw subspace.

Similarly, the defect manifold is generally not a screw subspace, since $\mathcal{D} = \$_6 - (\mathcal{V}_f^{\text{ray}} \cup \mathcal{W}_c)$, although [24, 37] both claim that the defect is a subspace. For example, the SAR manipulator in frame 2 has twist of freedom and wrench of constraint basis sets of

$$[{}^2\mathcal{V}_f^{\text{ray}}]_b = \Delta^2 J = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad [{}^2\mathcal{W}_c]_b = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \tag{6.72}$$

so that the radical basis set is

$$[{}^2\mathcal{R}]_b = \{[0,\ 0,\ 0,\ 0,\ 0,\ 1]^\tau\}\ . \tag{6.73}$$

The defect manifold contains all screws

$$[{}^2\mathcal{D}]_b = \{[\beta_x,\ \beta_y,\ \gamma,\ \delta_x,\ \delta_y,\ \delta_z]^\tau\}\ , \tag{6.74}$$

with nonzero γ. This is not a subspace, although [24] claims that a basis can be selected for the defect, $[{}^2\mathcal{D}]_b = \{[0,\ 0,\ \gamma,\ 0,\ 0,\ 0]^\tau\}$.

In frame 3, the SAR manipulator has twist of freedom and wrench of constraint basis sets of

$$[{}^3\mathcal{V}_f^{\text{ray}}]_b = \left\{ \begin{bmatrix} 0\\0\\0\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\d_3\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\1 \end{bmatrix} \right\},\quad [{}^3\mathcal{W}_c]_b = \left\{ \begin{bmatrix} 0\\0\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} \frac{-1}{d_3}\\0\\0\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\0\\0 \end{bmatrix} \right\}, \tag{6.75}$$

so that the radical basis set is empty, *i.e.,* $[{}^3\mathcal{R}]_b = \emptyset$. The defect manifold is also empty for the SAR manipulator in frame 3.

It is apparent now that the decomposition theory of [24, 37] is not unique and the claims made are generally invalid. Therefore a new technique for screw and wrench space decomposition is presented in the next section and the results of the previous sections of this chapter are tied together.

6.6 Space Decomposition at Decouple Point

In Section 6.2, it was shown that in some cases the twist space can be decomposed uniquely via a (Euclidean) direct sum decomposition, (see (6.53)) and in other cases not. In this section, the conditions for which this decomposition is possible are found.

When the wrenches of constraint are put in column-reduced echelon form, $[\mathcal{W}_c]_b E_w$, some of the columns may appear unitless. Since wrenches are screws, unitless columns will only exist in columns that have zeros in the force or moment positions. Each unitless column of $[\mathcal{W}_c]_b E_w$ represents one of the following two types of wrenches: the wrench is a pure force, *i.e.,*

$$W_{force} = [f_x,\ f_y,\ f_z,\ 0,\ 0,\ 0]^{\tau} \ , \tag{6.76}$$

or the wrench is a pure moment with respect to a frame on the wrench (screw) axis, *i.e.,*

$$W_{moment} = [0,\ 0,\ 0,\ n_x,\ n_y,\ n_z] \ . \tag{6.77}$$

Group these apparently unitless columns into $[\mathcal{W}_c^z]_b E_w$, the *wrenches of constraint with either zero force or zero moment.* The columns of $[\mathcal{W}_c]_b E_w$ that are not unitless are called the nonzero force and nonzero moment wrenches of constraint, $[\mathcal{W}_c^{nz}]_b E_w$.

If $[\mathcal{W}_c^z]_b E_w = [\mathcal{W}_c]_b E_w$, then the manipulator twist space decouples as shown in Theorem 12 below.

Theorem 12

$${}^i\mathcal{V} = {}^i\mathcal{V}_f \oplus {}^i\mathcal{V}_{nf} \Longleftrightarrow {}^i\mathcal{W}_c = {}^i\mathcal{W}_c^z$$

Proof

First prove that, in a given frame, there exists a direct sum decomposition of $\mathcal{V}$ if $\mathcal{W}_c = \mathcal{W}_c^z$; and then prove that, in a given frame, if there is a direct sum decomposition of $\mathcal{V}$, then $\mathcal{W}_c = \mathcal{W}_c^z$.

If $\mathcal{W}_c = \mathcal{W}_c^z$, the column-reduced echelon form basis vectors of $[\mathcal{W}_c]_b E_w$ have no units and can therefore be used for a basis of $\mathcal{V}_i$. But since the dimension of $\mathcal{W}_c$ plus the dimension of $\mathcal{V}_f$ is six and $\mathcal{W}_c = \mathcal{W}_c^z$, then $\mathcal{V}_{nf} = \mathcal{V}_i$. Therefore $[\mathcal{V}_{nf}]_b E_v = [\mathcal{W}_c^z]_b E_w$. This proves one half of the theorem.

The second half of the theorem is proven as follows. If the decomposition $\mathcal{V} = \mathcal{V}_f \oplus \mathcal{V}_{nf}$ is assumed, then the projection involved is Euclidean, *i.e.,*

$$\mathcal{V}_f = \text{Range}[JJ^{\dagger}] = \text{Range}[J] \ , \tag{6.78}$$

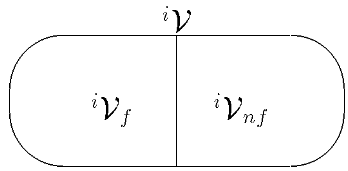

Figure 6.1. Decomposition of the twist space in frame i into decoupled subspaces.

and

$$\mathcal{V}_{nf} = \mathrm{Null}[JJ^\dagger] = \mathrm{Null}[J^\dagger] = \mathrm{Null}[J^\tau] \quad , \tag{6.79}$$

where $J^\dagger$ must be physically consistent from the assumption. But $\mathcal{W}_c = \mathrm{Null}[J^\tau]$ by definition. Since $\mathrm{Null}[J^\tau]$ can be interpreted as both a twist (of nonfreedom) and a wrench (of constraint), then $\mathcal{W}_c = \mathcal{W}_c^z$.

■

The twist space decomposition, when possible, is shown schematically in Figure 6.1. Conditions for this decomposition are given in Theorem 12 above and Theorem 13 below.

The above proof leads to a corollary that a subspace, $\mathcal{V}_s$, of $\mathcal{V}$ containing the twists of freedom, $\mathcal{V}_s \supseteq \mathcal{V}_f$, always has a direct sum decomposition $\mathcal{V}_s = \mathcal{V}_f \oplus \mathcal{V}_i$, *i.e.,*

$$[\mathcal{V}_i]_b E_v = [\mathcal{W}_c^z]_b E_w \quad , \tag{6.80}$$

where $\mathcal{V}_i$ does not exist (is empty) if there are no wrenches of constraint with zero force or zero moment in the chosen frame.

Corollary 5

$${}^i\mathcal{V}_s = {}^i\mathcal{V}_f \oplus {}^i\mathcal{V}_i \tag{6.81}$$

where ${}^i\mathcal{V}_f \subseteq {}^i\mathcal{V}_s \subseteq {}^i\mathcal{V}$.

Proof

If ${}^i\mathcal{W}_c^z = {}^i\mathcal{W}_c$, then the proof of this corollary is identical to the proof of Theorem 12 and ${}^i\mathcal{V}_s = {}^i\mathcal{V}$. Otherwise, if ${}^i\mathcal{W}_c^z \subset {}^i\mathcal{W}_c$, then the proof again follows the reasoning of the proof of Theorem 12, although the dimensions of the space ${}^i\mathcal{V}_s$ is reduced from 6 (the dimensions of ${}^i\mathcal{V}$) to $\mathrm{Dim}[\mathcal{V}_f] + \mathrm{Dim}[{}^i\mathcal{W}_c^z]$.

■

To continue this discussion of twist space decomposition, separate the twists of freedom into linear velocities of freedom and angular velocities of freedom, and separate the wrenches of constraint into forces of constraint and moments of constraint,

$$V_f = \begin{bmatrix} v_f \\ \omega_f \end{bmatrix} \quad , \quad W_c = \begin{bmatrix} f_c \\ n_c \end{bmatrix} \quad . \tag{6.82}$$

In a given frame i, if all ${}^if_c$ are orthogonal to all ${}^iv_f$ and all ${}^in_c$ are orthogonal to all ${}^i\omega_f$, then the manipulator decouples and the twist space can be uniquely decomposed into twists of freedom and twists of nonfreedom subspaces.

Theorem 13

$${}^i\mathcal{V} = {}^i\mathcal{V}_f \oplus {}^i\mathcal{V}_{nf} \Longleftrightarrow \begin{cases} {}^if_c \odot {}^iv_f \;= 0, \; \forall\, {}^if_c, {}^iv_f \\ {}^in_c \odot {}^i\omega_f = 0, \; \forall\, {}^in_c, {}^i\omega_f \end{cases} \quad .$$

Proof

Assume ${}^i\mathcal{V} = {}^i\mathcal{V}_f \oplus {}^i\mathcal{V}_{nf}$ and remember from (1.35) that ${}^iV_f \circ {}^i\mathcal{W}_c = 0$. From Theorem 12, ${}^i\mathcal{W}_c = {}^i\mathcal{W}_c^z$, which implies that either ${}^if_c = 0$ or ${}^in_c = 0$ for each wrench in ${}^i\mathcal{W}_c$. In the case ${}^if_c = 0$,

$${}^iV_f \circ {}^iW_c = {}^i\omega_f \odot {}^in_c = 0 \quad , \tag{6.83}$$

and the right-hand side of the theorem is proven. In the case ${}^in_c = 0$, then

$${}^iV_f \circ {}^iW_c = {}^iv_f \odot {}^if_c = 0 \quad , \tag{6.84}$$

completing the proof that the right-hand side of the theorem follows from the left-hand side.

The other direction of the proof proceeds as follows. The right-hand side of the theorem implies that ${}^i\mathcal{W}_c = {}^i\mathcal{W}_c^z$, and then the proof of Theorem 12 will suffice.

■

For example, the wrenches of constraint in column-reduced echelon form of the PR virtual manipulator of Figure 4.1, expressed in frame 2 is

$$\left[{}^2\mathcal{W}_c\right]_b E_w = \left[\mathrm{Null}[{}^2J^\tau]\right]_b E_w = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{6.85}$$

where (4.20) gives the Jacobian of this manipulator. The conditions on the right hand side of Theorem 12 are satisfied since the above matrix is also $[{}^2\mathcal{W}_c^z]_b E_w$. The conditions on the right hand side of Theorem 13 are also met since ${}^2f_c \odot {}^2v_f = 0$ and ${}^2n_c \odot {}^2\omega_f = 0$ for all ${}^2f_c$, ${}^2v_f$, ${}^2n_c$, and ${}^2\omega_f$. Therefore, both Theorem 12 and Theorem 13 tell us that the decomposition of the twist space into unique disjoint subspaces is valid in this frame.

The Jacobian of the PR manipulator expressed in the translated frame t was given in (4.22). The wrenches of constraint in column-reduced echelon form are

$$\left[{}^t\mathcal{W}_c\right]_b E_w = \begin{bmatrix} \frac{-1}{p_y} & 0 & 0 & \frac{p_x}{p_y} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} , \tag{6.86}$$

where $p_y \neq 0$. If $p_y = 0$, the first column of $[{}^t\mathcal{W}_c]_b E_w$ is replaced by $[0,\ 1/p_x,\ 0,\ 0,\ 0,\ 1]^\tau$ and the last column by $[1,\ 0,\ 0,\ 0,\ 0,\ 0]^\tau$.

The requirement of the right hand side of Theorem 12 is violated by the above $[{}^{t}\mathcal{W}_c]_b E_w$. Also, both conditions on the right hand side of Theorem 13 are violated by the wrench in the first column of (6.86).

Theorem 12 and Theorem 13 lead to a similar unique decomposition of the wrench space. The wrench space can sometimes be split into two disjoint subspaces, the wrenches of constraint and the wrenches of nonconstraint, ${}^{i}\mathcal{W}_{nc}$. But first define a subspace $\mathcal{V}_f^z$ in a manner similar to the definition of $\mathcal{W}_c^z$, *i.e.,* $[\mathcal{V}_f^z]_b E_v$ are the *twists of freedom with either zero linear velocity or zero angular velocity.* This leads to Theorem 14 below.

Theorem 14

$${}^{i}\mathcal{W} = {}^{i}\mathcal{W}_{nc} \oplus {}^{i}\mathcal{W}_c \Longleftrightarrow \mathcal{V}_f = \mathcal{V}_f^z$$

Proof

First prove that there exists a direct sum decomposition of $\mathcal{W}$ if $\mathcal{V}_f = \mathcal{V}_f^z$; and then prove that if there is a direct sum decomposition of $\mathcal{W}$, then $\mathcal{V}_f = \mathcal{V}_f^z$.

If $\mathcal{V}_f = \mathcal{V}_f^z$, the column-reduced echelon form basis vectors of $[\mathcal{V}_f]_b E_w$ have no units and can therefore be used for a basis of $\mathcal{W}_d$. But since the dimension of $\mathcal{V}_f$ plus the dimension of $\mathcal{W}_c$ is six and $\mathcal{V}_f = \mathcal{V}_f^z$, then $\mathcal{W}_{nc} = \mathcal{W}_d$. Therefore $[\mathcal{W}_{nc}]_b E_w = [\mathcal{V}_f^z]_b E_v$. This proves one half of the theorem.

The second half of the theorem is proven as follows. If the decomposition $\mathcal{W} = \mathcal{W}_c \oplus \mathcal{W}_{nc}$ is assumed, then the projection involved is Euclidean, *i.e.,*

$$\mathcal{W}_c = \mathrm{Null}[JJ^{\dagger}] = \mathrm{Null}[J^{\tau}] \ , \tag{6.87}$$

and

$$\mathcal{W}_{nc} = \mathrm{Range}[JJ^{\dagger}] = \mathrm{Range}[J] \ , \tag{6.88}$$

where $J^{\dagger}$ must be physically consistent from the assumption. But $\mathcal{V}_f = \mathrm{Range}[J]$ by definition. Since Range$[J]$ can be interpreted as both a wrench (of nonconstraint) and a twist (of freedom), then $\mathcal{V}_f = \mathcal{V}_f^z$.

■

Finally, Theorem 15 below shows the equivalence of the decomposition of the twist and wrench spaces when $J^\dagger$ is physically consistent *i.e.,* the unique Euclidean decomposition of the twists space results in the unique Euclidean decomposition of the wrench space, and vice-versa.

<u>*Theorem 15*</u> *If $J^\dagger$ is physically consistent, the following are equivalent statements:*

$$ {}^i\mathcal{V}_f = {}^i\mathcal{V}_f^z \tag{6.89} $$

$$ {}^i\mathcal{W}_c = {}^i\mathcal{W}_c^z \tag{6.90} $$

$$ {}^i\mathcal{W} = {}^i\mathcal{W}_{nc} \oplus {}^i\mathcal{W}_c \tag{6.91} $$

$$ {}^i\mathcal{V} = {}^i\mathcal{V}_f \oplus {}^i\mathcal{V}_c \tag{6.92} $$

$$ {}^i\mathcal{V} = {}^i\mathcal{W} = {}^i\mathcal{V}_f \oplus {}^i\mathcal{W}_c = \text{Range}[J] \oplus \text{Null}[J^T] \ . \tag{6.93} $$

<u>Proof</u>

If $\mathcal{V}_f = \mathcal{V}_f^z \Leftrightarrow \mathcal{W}_c = \mathcal{W}_c^z$, Theorem 12 and Theorem 14 can be used to prove the equivalence of the rest of the statements. From Theorem 12,

$$ \mathcal{W}_c = \mathcal{W}_c^z \Leftrightarrow \mathcal{V} = \mathcal{V}_f \oplus \mathcal{V}_{nf} = \mathcal{V}_f \oplus \mathcal{W}_c \ . \tag{6.94} $$

From Theorem 14,

$$ \mathcal{V}_f = \mathcal{V}_f^z \Leftrightarrow \mathcal{W} = \mathcal{W}_c \oplus \mathcal{W}_{nc} = \mathcal{W}_c \oplus \mathcal{V}_f \ . \tag{6.95} $$

Since the right-hand-side decomposition of these two equations are identical, (6.89) and (6.90) are equivalent statements.

■

If the twist and wrench screw spaces are uniquely decomposable in a chosen frame, then a rotation of the frame of expression on the disjoint subspaces will preserve disjointedness since ${}^iG_j = {}^iA_j$. But a translation of the frame of expression will not preserve the decomposition of the subspaces. In fact, only special manipulators have the two unique subspaces (twists of constraint and wrenches of freedom) for twist and wrench space decompositions in all configurations. (These manipulators will be discussed in Section 6.7.) Generally, the set of twists that a manipulator cannot achieve, $\mathcal{V}_{nf}$, is not a subspace of twists so no unique $\mathcal{V}_{nf}$ can be found; and generally, the set of wrenches that a manipulator can apply, $\mathcal{W}_{nc}$, is not a subspace of wrenches so no unique $\mathcal{W}_{nc}$ can be found.

The SCARA and the planar RRR manipulator discussed earlier are special manipulators that decouple the twist and wrench spaces into two disjoint subspaces in *all* frames of expression. For the SCARA manipulator in a frame arbitrarily translated from frame 2, the column-reduced echelon form twists of freedom and the column-reduced echelon form wrenches of constraint are

$$\left[{}^t\mathcal{V}_f\right]_b E_v = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^T \quad \left[{}^t\mathcal{W}_c\right]_b E_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}^T . \tag{6.96}$$

Since each of the column-reduced echelon form twists of freedom have zero linear velocity or zero angular velocity, the manipulator decouples. It also decouples since each of the column-reduced echelon form wrenches of constraint have zero force. Since both of the constraint wrenches have zero force, this manipulator can apply a force to the environment in any direction as long as the manipulator is not in a singular configuration.

The terms decouple frame and decouple point are defined in Section 3.2 and Section 4.1.1, respectively. The pseudo-inverse of the manipulator Jacobian in a frame located at a decouple point (a decouple frame) is physically consistent. Some new meaning of decouple points can now be presented.

Theorem 15 is based on the condition that $J^\dagger$ is physically consistent, *i.e.,* the frame of expression is located at a decouple point. All of the statements in this theorem are therefore the requirements necessary for a manipulator space, with respect to a particular frame, to decouple. If the frame of expression is at a decouple point, the twist and wrench spaces decouple identically as shown in (6.43) and (6.93).

Raibert and Craig [51] define a "constraint frame" as a frame in which the natural and "orthogonal" artificial constraints can be independently specified. A constraint frame or a compliant frame [2] is a frame in which the twist and wrench spaces decouple entirely into subspaces, and therefore twists and wrenches may be uniquely decomposed into constraint and freedom components. For the SCARA and the planar RRR manipulators, all frames are compliant frames.

The author of this paper prefers the term *decouple point* to describe a point at which a frame can be placed that will allow the twist and wrench spaces to be uniquely decomposed. This is also a point at which the pseudo-inverse is physically consistent. In fact, at a decouple point, the fundamental theorem of algebra for commensurate systems is meaningful for this noncommensurate system. As was shown in the Chapter 4, any rotations of the frame at this point will not affect the decoupled nature of the spaces.

When the frame of expression is not located at a decouple point, the twist and wrench spaces cannot be uniquely decomposed by a direct sum. But, a part of the twist or wrench spaces may be uniquely decomposable so that

$$\begin{aligned} \text{Subspace}[{}^i\mathcal{V}] &= {}^i\mathcal{V}_f \oplus {}^i\mathcal{V}_i = {}^i\mathcal{V}_f \oplus {}^i\mathcal{W}_c^z \neq {}^i\mathcal{V} & (6.97) \\ \text{Subspace}[{}^i\mathcal{W}] &= {}^i\mathcal{W}_c \oplus {}^i\mathcal{W}_d = {}^i\mathcal{W}_c \oplus {}^i\mathcal{V}_f^z \neq {}^i\mathcal{W}\ . & (6.98) \end{aligned}$$

For any frame i, a wrench coordinate transformation ${}^iA^{t,i}$ exist that will convert any single wrench of constraint with nonzero force and nonzero moment to a wrench with a zero moment and the same force. This particular wrench coordinate transformation consists of a translation vector of

$$p = \frac{n \times f}{|f|^2} \tag{6.99}$$

and no rotation. Note that this transformation will also generally convert other wrenches that had zero moments to wrenches with nonzero moments.

Therefore, for all manipulators with Jacobian of rank less then six (*i.e.,* a non-empty wrench of constraint subspace), there exists a frame that makes at least one of the constraint wrenches into an element of $\mathcal{W}_c^z$, and thus $\mathcal{W}_c^z \neq \emptyset$ in some frame.

For example, a P50 manipulator in frame 3 coordinates has the Jacobian and column-reduced echelon form wrench of constraint basis of

$$^3J = \begin{bmatrix} 0 & a_2 s_3 & 0 & 0 & 0 \\ 0 & a_3 + a_2 c_3 & a_3 & 0 & 0 \\ -a_2 c_2 - a_3 c_{2+3} & 0 & 0 & 0 & 0 \\ s_{2+3} & 0 & 0 & 0 & s_4 \\ c_{2+3} & 0 & 0 & 0 & -c_4 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad [^3\mathcal{W}_c]_b E_w = \begin{bmatrix} 0 \\ 0 \\ \frac{s_{2+3+4}}{s_4(a_2 c_2 + a_3 c_{2+3})} \\ \frac{c_4}{s_4} \\ 1 \\ 0 \end{bmatrix} . \tag{6.100}$$

Note that frame 3 is not a decouple frame. But (6.99) can be used with (1.16) to find a frame where the manipulator does decouple,

$$p = \left[\frac{(a_2 c_2 + a_3 c_{2+3}) s_4}{s_{2+3+4}},\ -\frac{(a_2 c_2 + a_3 c_{2+3}) c_4}{s_{2+3+4}},\ 0 \right]^\tau \tag{6.101}$$

$$^t\mathcal{W}_c = {}^3A^{t,3}[^3\mathcal{W}_c]_b E_w = \left[0,\ 0,\ \frac{s_{2+3+4}}{(a_2 c_2 + a_3 c_{2+3}) s_4},\ 0,\ 0,\ 0 \right]^\tau . \tag{6.102}$$

The physically consistent determinant of $J^\tau J$ in frame t is

$$\mathrm{Det}[{}^tJ^\tau\ {}^tJ] = (a_2 a_3 s_3 s_{2+3+4})^2 . \tag{6.103}$$

A non-planar RRR manipulator with Denavit-Hartenberg parameters given in Table 6.1 has a frame 2 Jacobian and column-reduced echelon form wrenches of

Table 6.1. D-H parameters for a non-planar RRR manipulator.

Joint Type	d	a	θ	α
R	0	a_1	θ_1	$\pi/2$
R	d_2	a_2	θ_2	$\pi/2$
R	0	0	θ_3	0

constraint basis of

$$
{}^2J = \begin{bmatrix} d_2c_2 & 0 & 0 \\ -a_1 - a_2c_2 & 0 & 0 \\ d_2s_2 & -a_2 & 0 \\ s_2 & 0 & 0 \\ 0 & 1 & 0 \\ -c_2 & 0 & 1 \end{bmatrix} , \quad \left[{}^2\mathcal{W}_c\right]_b E_w = \begin{bmatrix} -\frac{s_2}{a_2c_2} & -\frac{s_2}{d_2c_2} & \frac{a_1+a_2c_2}{d_2c_2} \\ 0 & 0 & 1 \\ \frac{1}{a_2} & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^\tau . \tag{6.104}
$$

Since no single wrench coordinate transformation will convert both of the first two columns of $[{}^2\mathcal{W}_cE_w]_b$ to a form that satisfies the right hand side of Theorem 12, this manipulator has no frame at which the twist and wrench spaces decouple. Wrench coordinate transformations will not affect column 3, a pure force, from remaining a pure force. Either column 1 or column 2 can be made into a zero force wrench of constraint given an appropriate wrench coordinate transformation with translation calculated from (6.99). In this case $\mathcal{W}_c^z$ will be composed of two wrenches instead of the one in frame 2.

As promised in Section 3.2, it will now be shown that the physical consistency of the determinant of $J^\tau J$ in a particular frame assures that $J^\dagger$ in that frame is physically consistent for all configurations, and thus the frame of expression is located at a decouple point. Let E_v be defined as previously, such that it puts a twist basis set into column-reduced echelon form. So $JE_v = [V_f]_bE_v$, is the column-reduced echelon form of the twists of freedom. From the previous results we know that $[V_f]_bE_v = [V_f^z]_bE_v$ when $J^\dagger$ is physically consistent.

The determinant of $J^\tau J$ in a chosen frame can be expanded as follows:

$$
\mathrm{Det}[J^\tau J] = \mathrm{Det}[E_v^{-\tau}E_v^\tau J^\tau JE_vE_v^{-1}] = \frac{\mathrm{Det}[(JE_v)^\tau(JE_v)]}{\mathrm{Det}[E_v]^2} , \tag{6.105}
$$

since $\mathrm{Det}[E_v] = \mathrm{Det}[E_v^\tau]$. The units of each element in a given row of E_v are identical and therefore the determinant of E_v is physically consistent. Then from (6.105), $\mathrm{Det}[J^\tau J]$ is physically consistent if and only if $\mathrm{Det}[(JE_v)^\tau(JE_v)]$ is physically consistent. The matrix $[\mathcal{V}_f]_b E_v = JE_v$ has no physical units if $J^\dagger$ is physically consistent and therefore $\mathrm{Det}[(JE_v)^\tau(JE_v)]$ is physically consistent if $J^\dagger$ is physically consistent. Hence, the frame of expression is a decouple point and $\mathrm{Det}[J^\tau J]$ is physically consistent if and only if $J^\dagger$ is physically consistent.

For example, the SCARA manipulator in frame 2 coordinates has

$$
{}^2JE_v = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad , \quad E_v = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{a_1 s_2} \\ 0 & 0 & \frac{1}{a_2} & \frac{a_2 + a_1 c_2}{a_1 a_2 s_2} \\ 1 & 0 & -\frac{1}{a_2} & \frac{a_1 c_2}{a_1 a_2 s_2} \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad . \tag{6.106}
$$

The determinants in frame 2 coordinates, $\mathrm{Det}[{}^2(J^\tau J)] = a_1 a_2 s_2^2$ and $\mathrm{Det}[({}^2JE_v)^\tau({}^2JE_v)] = 1$, are both physically consistent and thus frame 2 is a decouple frame for the SCARA manipulator. The determinant of E_v is $-1/(a_1 a_2 s_2)$.

For another example, the RRRP-2 manipulator expressed in frame 2 coordinates has

$$
{}^2JE_v = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{a_1 - a_2 c_2}{c_2} & 0 & 0 \\ 0 & \frac{s_2}{c_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad , \quad E_v = \begin{bmatrix} 0 & \frac{1}{c_2} & 0 & 0 \\ 0 & 0 & \frac{1}{a_2} & \frac{c_3}{a_2 s_3} \\ 1 & 0 & \frac{-1}{a_2} & \frac{-c_3}{a_2 s_3} \\ 0 & 0 & 0 & \frac{1}{s_3} \end{bmatrix} \quad . \tag{6.107}
$$

The determinants in frame 2 coordinates,

$$
\mathrm{Det}[{}^2(J^\tau J)] = (a_2 s_3)^2 (1 + a_1^2 + a_2^2 c_2^2 + 2 a_1 a_2 c_2) \tag{6.108}
$$

and

$$
\mathrm{Det}[({}^2JE_v)^\tau({}^2JE_v)] = \frac{1}{c_2^2}(1 + a_1^2 + a_2^2 c_2^2 + 2 a_1 a_2 c_2) \quad , \tag{6.109}
$$

are both physically inconsistent and thus frame 2 is not a decouple frame for the RRRP-2 manipulator. The determinant of E_v is $-1/(a_2 c_2 s_3)$.

Tests can now be clearly stated to determine the conditions for decouple points. A test to determine if a manipulator decouples for all configurations with origins at the origin of frame i is

$$\text{physically consistent Det}[{}^iJ^\tau\ {}^iJ)] \text{ for all configurations.} \tag{6.110}$$

A test to determine if a manipulator decouples for all configurations at every point is

$$\text{physically consistent Det}\left[\left({}^iG^{t,i}\ {}^iJ\right)^\tau {}^iG^{t,i}\ {}^iJ\right] \text{ for all configurations, or} \tag{6.111}$$

$$\text{physically consistent Det}\left[{}^iJ^\tau \left({}^iG^{t,i}\right)^\tau\ {}^iG^{t,i}\ {}^iJ\right] \text{ for all configurations,} \tag{6.112}$$

where ${}^iG^{t,i}$ is the general translation matrix given by (1.4) with no rotation ($R = I_3$), and i is some convenient frame. (Midframe Jacobians are always simpler symbolically than end-frame or base-frame Jacobians [13].) Note that the matrix product $({}^iG^{t,i})^\tau\ {}^iG^{t,i}$ in (6.112) is physically inconsistent.

Of course the above test of (6.111) and (6.112) or any of the other tests for decouple points can be used with a general translation matrix to determine the various conditions for decouple points. Perhaps the simplest test to find a manipulator's decouple points is to determine the conditions (if any) for which ${}^iG^{t,i}\ {}^iJE_v^{GJ} = \mathcal{V}_f^z$, where E_v^{GJ} is the matrix that puts ${}^iG^{t,i}\ {}^iJE_v^{GJ}$ in column-reduced echelon form.

The final part of the twist space is the twist defect manifold, $\mathcal{V}_{dm}$,

$$\mathcal{V}_{dm} = \mathcal{V} - \mathcal{V}_f - \mathcal{V}_i \ . \tag{6.113}$$

Similarly, the final part of the wrench space is the wrench defect manifold, $\mathcal{W}_{dm}$,

$$\mathcal{W}_{dm} = \mathcal{W} - \mathcal{W}_c - \mathcal{W}_d \ . \tag{6.114}$$

A nonempty twist or wrench defect manifold is never a subspace. Both the twist and wrench defect manifolds are empty if the frame of expression is a decouple frame.

To summarize, several tests to determine if a point (at which the coordinates of a manipulator are expressed) is a decouple point are as follows:

- for all configurations the matrix ${}^{i}J^{\dagger}$ is physically consistent,
- for all configurations the $\mathrm{Det}[{}^{i}J^{\tau}\ {}^{i}J]$ is physically consistent,
- for all configurations $[{}^{i}\mathcal{V}_f]_b E_v = [{}^{i}\mathcal{V}_f^z]_b E_v$, and
- for all configurations $[{}^{i}\mathcal{W}_c]_b E_w = [{}^{i}\mathcal{W}_c^z]_b E_w$.

6.7 Self-Reciprocal Manipulators

A class of manipulators are now introduced for which the twists of freedom are self-reciprocal, *i.e.,*

$$J^{\tau}\Delta J = [0]_{n,n} \quad . \tag{6.115}$$

Expanding (6.115), the self-reciprocal test becomes

$$J^{\tau}\Delta J = J_v^{\tau} J_{\omega} + J_{\omega}^{\tau} J_v = [0]_{n,n} \quad , \tag{6.116}$$

where

$$J = \begin{bmatrix} J_v \\ J_{\omega} \end{bmatrix} \tag{6.117}$$

and

$$J_v^{\tau} J_{\omega} = (J_{\omega}^{\tau} J_v)^{\tau} \quad . \tag{6.118}$$

Definition 7 A manipulator is *self-reciprocal for configuration q in frame i* if and only if ${}^{i}J(q)^{\tau}\Delta {}^{i}J(q) = [0]_{n,n}$.

Theorem 16 below shows that ${}^{i}J^{\tau}\Delta {}^{i}J = [0]_{n,n}$ is a frame independent characteristic.

Theorem 16 *If a manipulator is self-reciprocal for configuration q in frame i, then it is also self-reciprocal for configuration q in any other frame j.*

Proof

Assume ${}^{i}(J^{\tau}\Delta J) = [0]_{n,n}$, where $J = J(q)$, *i.e.,* the configuration dependency is assumed. From (1.4) and (1.12),

$$ {}^{j}J = {}^{j}G_{i}^{j,i}\,{}^{i}J = \begin{bmatrix} R & BR \\ 0 & R \end{bmatrix} \begin{bmatrix} {}^{i}J_{v} \\ {}^{i}J_{\omega} \end{bmatrix} = \begin{bmatrix} R^{i}J_{v} + BR^{i}J_{\omega} \\ R^{i}J_{\omega} \end{bmatrix} \quad . \tag{6.119} $$

Then

$$ {}^{j}J^{\tau}\Delta^{j}J = {}^{i}(J_{v}^{\tau}J_{\omega} + J_{\omega}^{\tau}J_{v}) + {}^{i}(J_{\omega}R^{\tau}(B^{\tau} + B)RJ_{\omega}) \quad . \tag{6.120} $$

But by (1.8), $B^{\tau} + B = [0]_{3,3}$, so that using (6.116) results in

$$ {}^{j}J^{\tau}\Delta^{j}J = {}^{i}(J_{v}^{\tau}J_{\omega} + J_{\omega}^{\tau}J_{v}) = {}^{i}J^{\tau}\Delta^{i}J = [0]_{n,n} \quad . \tag{6.121} $$

■

This theorem leads to the definition of self-reciprocal manipulators below.

Definition 8 A *manipulator is self-reciprocal* if and only if $J(q)^{\tau}\Delta J(q) = [0]_{n,n}$ for all configurations q.

A subset of the self-reciprocal manipulators are the manipulators for which both terms in the summation of (6.116) are zero, *i.e.,*

$$ J_{v}^{\tau}J_{\omega} = J_{\omega}^{\tau}J_{v} = [0]_{n,n} \quad . \tag{6.122} $$

There are two types of manipulators that satisfy (6.122): all prismatic-jointed manipulators and planar manipulators. Planar manipulators are defined below.

Definition 9 A *planar manipulator* will create only linear motion in a plane and angular motion perpendicular to that plane for all configurations.

Consequences of this definition for robots composed entirely of revolute and prismatic joints are given in Fact 4 below.

Fact 4 Planar manipulators have the following characteristics:

- The cross products of all prismatic joint axes are parallel.
- Revolute joint axes are parallel.
- All revolute joint axes are orthogonal to all prismatic joint axes.
- The cross products of all prismatic joint axes are parallel to all revolute joint axes.

Spherical manipulators are self-reciprocal but do not satisfy (6.122). The definition of spherical manipulators is given in the definition below.

Definition 10 A *spherical manipulator* will create linear motion of any fixed point in the tool-frame only in directions tangent to a sphere's surface. The sphere is fixed for every tool shape. There are no constraints on angular motion for spherical manipulators.

Consequences of this definition for robots composed entirely of revolute and prismatic joints are given in Fact 5 below.

Fact 5 Spherical manipulators have the following characteristics:

- They are composed entirely of revolute joints, *i.e.,* have no prismatic joints.
- All revolute joint axes intersect at a single point.

The self-reciprocal condition, $J^{\tau}\Delta J = [0]_{n,n}$, is valid for all spherical manipulators, but $J_v^{\tau} J_\omega = [0]_{n,n}$ only in a frame located at the intersection point of each spherical manipulator's joint axes. This is illustrated by the below example.

The RR manipulator is a spherical manipulator whose Jacobian, ${}^tJ$, in a frame arbitrarily translated from frame 2 equals

$$ {}^tJ = {}^2G^{t,2}\,{}^2J = \begin{bmatrix} p_y\kappa_1 - p_z c_2\sigma_1 & p_y \\ -p_x\kappa_1 + p_z\sigma_1 s_2 & -p_x \\ p_x c_2\sigma_1 - p_y\sigma_1 s_2 & 0 \\ \sigma_1 s_2 & 0 \\ c_2\sigma_1 & 0 \\ \kappa_1 & 1 \end{bmatrix} . \tag{6.123} $$

This results in $J^\tau\Delta J = [0]_{n,n}$, but

$$ {}^tJ_v^\tau J_\omega = -{}^tJ_\omega^\tau J_v = \begin{bmatrix} 0 & \sigma_1(-p_x c_2 + p_y s_2) \\ \sigma_1(p_x c_2 - p_y s_2) & 0 \end{bmatrix} . \tag{6.124} $$

The RR manipulator is self-reciprocal, but $J_v^\tau J_\omega = [0]_{n,n}$ for all configurations only at the axes intersection, *i.e.,* $p_x = p_y = 0$.

Similarly, the RRR spherical manipulator is self-reciprocal but

$$ {}^tJ_v^\tau J_\omega = -{}^tJ_\omega^\tau J_v = \begin{bmatrix} 0 & c_2p_x + p_z s_2 & -p_y s_2 \\ -c_2p_x - p_z s_2 & 0 & p_x \\ p_y s_2 & -p_x & 0 \end{bmatrix} . \tag{6.125} $$

If $p_x = p_y = p_z = 0$ in (6.125), then the frame is at the intersection of the joint axes and $J_v^\tau J_\omega = 0$.

In order to find the class of all RRR manipulators that are self-reciprocal, let us look at $J^\tau\Delta J$ in frame 2 for the general RRR manipulator,

$$ {}^2(J^\tau\Delta J) = \begin{bmatrix} 0 & a_1\sigma_1 & \gamma \\ -a_1\sigma_1 & 0 & -a_2\sigma_2 \\ \gamma & -a_2\sigma_2 & 0 \end{bmatrix} , \tag{6.126} $$

where

$$ \gamma = -a_1\kappa_2\sigma_1 - a_2\kappa_2c_2\sigma_1 - a_2\kappa_1\sigma_2 - a_1\kappa_1c_2\sigma_2 + d_2\sigma_1\sigma_2s_2 \ . \tag{6.127} $$

The conditions for which $J^\tau\Delta J = [0]_{n,n}$ in all configurations for the general RRR manipulator are

$$ a_1 = 0 \text{ and } a_2 = 0 \text{ and } (\sigma_1 = 0 \text{ or } \sigma_2 = 0 \text{ or } d_2 = 0) \tag{6.128} $$

or

$$\sigma_1 = 0 \text{ and } \sigma_2 = 0 \quad . \tag{6.129}$$

Each of these cases result in a manipulator that is either planar or spherical. If any of the set of conditions of (6.128) is valid, then the manipulator is spherical. If the conditions of (6.129) are valid, then the manipulator is planar.

The results are summarized in the facts below.

Fact 6 Self-reciprocal manipulators are:

- Only prismatic-jointed,
- Planar, or
- Spherical.

Fact 7 Entirely prismatic-jointed manipulators and planar manipulators decouple and have $J_v^\tau J_\omega = [0]_{n,n}$ in all frames.

Fact 8 Spherical manipulators are self-reciprocal. For these manipulators, $J_v^\tau J_\omega = -J_\omega^\tau J_v$. At the intersection of the revolute axes $J_v^\tau J_\omega = [0]_{n,n}$.

If a manipulator is self-reciprocal then

$$\gamma \text{Range}[\Delta J] \subseteq \mathcal{W}_c \quad , \tag{6.130}$$

for some scalar γ. This is derived from Theorem 2 and the resulting fact that $J^\tau W_c = [0]_{n,6-r}$ for any W_c and $J^\tau(\Delta J) = [0]_{n,n}$ for all configurations, where r is the column rank of J.

If the column rank of J is 3 then

$$\gamma \text{Range}[\Delta J] = \mathcal{W}_c \quad , \tag{6.131}$$

since the rank of J plus the rank of $\mathcal{W}_c$ is always equal to six. Therefore, the maximum number of independent joints for a reciprocal manipulator is three. Manipulators with more than three joints may be self-reciprocal only if they are redundant, *i.e.,* the rank of J is less than or equal to 3.

The (i,i)-th term of $J^\tau \Delta J$ is always zero for any manipulator since by Theorem 11 all columns of a manipulator Jacobian are self-reciprocal, *i.e.,*

$$(J^\tau \Delta J)_{(i,i)} = (J)_{(\cdot,i)} \odot (\Delta J)_{(\cdot,i)} = 2(J_v)_{(\cdot,i)} \odot (J_\omega)_{(\cdot,i)} = 0 \tag{6.132}$$

so that

$$(J_v)_{(\cdot,i)} \odot (J_\omega)_{(\cdot,i)} = 0 \quad . \tag{6.133}$$

Let the i-th column of J_v be represented by v_i and the i-th column of J_ω be represented by ω_i. The twist due to the i-joint is therefore $V_i = [v_i,\ \omega_i]^\tau$ and (6.133) becomes

$$v_i \odot \omega_i = v_i^\tau \omega_i = 0 \quad , \tag{6.134}$$

regardless of the frame of expression or the configuration of the manipulator.

For planar manipulators, the conditions in Fact 4 make the further requirements that

$$v_i \odot \omega_j \;=\; 0 \;\; , \forall\ i,\ j \tag{6.135}$$

$$\omega_i \times \omega_j \;=\; 0 \;\; , \forall\ i,\ j \tag{6.136}$$

$$\frac{v_i \times v_j}{|v_i \times v_j|} \;=\; \begin{cases} \frac{v_k \times v_l}{|v_k \times v_l|} \\ \text{or} \\ 0 \end{cases} \quad , \forall\ i,\ j,\ k,\ l. \tag{6.137}$$

This leads to Theorem 17 below.

Theorem 17 *Planar manipulators decouple in every frame.*

Proof

By the definition of planar motion, there exists a frame such that the linear motion is in the xy-plane and the angular motion is about the z-axis for all configurations.

If the rank of the Jacobian is 3, then in this frame the twist of freedom are

$$\mathcal{V}_f = \text{Range}[B_f] \ , \quad B_f = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \ . \tag{6.138}$$

Then translation in the z direction and rotation about the x and y axis are inaccessible twists,

$$\mathcal{V}_i = \text{Range}[B_i] \ , \quad B_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ . \tag{6.139}$$

These twists constitute a subspace. Since each of the inaccessible twists have either $v = 0$ or $\omega = 0$, they may appropriately be interpreted as wrenches of constraint. Thus, by Theorem 12, all planar manipulators with Jacobian of rank 3 decouple in this frame.

If the manipulator has less than three joints (or the rank of J is less than three), then one or more of the freedom twists above will become inaccessible twists. The new inaccessible twist or twists may also be interpreted as wrenches of constraint since each of the possible twist of freedom in (6.138) also meet the requirement of either $v = 0$ or $\omega = 0$. Thus, all planar manipulator decouple in this frame.

A translation of the frame does not affect the inaccessible twists for planar manipulators. It has been previously shown that frame rotations have no affect on whether a manipulator decouples or not. Therefore, planar manipulators decouple in all frames.

■

Since planar manipulators decouple at every point, then $J^{\dagger}$ is physically consistent regardless of the frame of expression. When using the pseudo-inverse solution of the inverse velocity problem some researchers have therefore termed a solution as "optimal" when in fact the solution is still only "optimal" with respect to a physically inconsistent Euclidean norm. This "optimal" result is a generalized-inverse solution using identity metrics in the "optimal" frame. Applying the pseudo-inverse solution in another frame will give a different "optimal" solution corresponding to an identity metric in this new frame.

For example, the planar RRR manipulator is solved below in frame 2 coordinates and in a frame arbitrarily translated from frame 2. The desired twist is

$$ {}^2V = \begin{bmatrix} 1\frac{\mathrm{m}}{\mathrm{s}} \\ 2\frac{\mathrm{m}}{\mathrm{s}} \\ 3\frac{\mathrm{m}}{\mathrm{s}} \\ 4\frac{\mathrm{rad}}{\mathrm{s}} \\ 5\frac{\mathrm{rad}}{\mathrm{s}} \\ \frac{\mathrm{m}}{\mathrm{s}} \end{bmatrix} \quad , \quad {}^tV = {}^2G^{t,2}\,{}^2V = \begin{bmatrix} 1\frac{\mathrm{m}}{\mathrm{s}} + \frac{6p_y}{\mathrm{s}} - \frac{5p_z}{\mathrm{s}} \\ 2\frac{\mathrm{m}}{\mathrm{s}} - \frac{6p_x}{\mathrm{s}} + \frac{4p_z}{\mathrm{s}} \\ 3\frac{\mathrm{m}}{\mathrm{s}} + \frac{5p_x}{\mathrm{s}} - \frac{4p_y}{\mathrm{s}} \\ 4\frac{\mathrm{rad}}{\mathrm{s}} \\ 5\frac{\mathrm{rad}}{\mathrm{s}} \\ 6\frac{\mathrm{m}}{\mathrm{s}} \end{bmatrix} \quad . \tag{6.140} $$

The following parameter values have been selected:

$$ a_1 = a_2 = 1\mathrm{m}\ ,\ \ \theta_1 = \pi/4\mathrm{rad}\ ,\ \ \theta_2 = \pi/6\mathrm{rad}\ ,\ \ \theta_3 = \pi/7\mathrm{rad}\ \ . \tag{6.141} $$

The pseudo-inverse solutions in each of these frames are

$$ \dot{q}_{s2} = {}^2J^{\dagger}\ {}^2V = \begin{bmatrix} 2\frac{\mathrm{rad}}{\mathrm{s}} \\ -1.732\frac{\mathrm{rad}}{\mathrm{s}} \\ 5.732\frac{\mathrm{rad}}{\mathrm{s}} \end{bmatrix} \quad , \quad \dot{q}_{st} = {}^tJ^{\dagger}\ {}^tV = \begin{bmatrix} 2\frac{\mathrm{rad}}{\mathrm{s}} - \frac{10p_z}{\mathrm{ms}} \\ -1.732\frac{\mathrm{rad}}{\mathrm{s}} + \frac{22.660p_z}{\mathrm{ms}} \\ 5.732\frac{\mathrm{rad}}{\mathrm{s}} - \frac{12.660p_z}{\mathrm{ms}} \end{bmatrix} \quad . \tag{6.142} $$

For any nonzero p_z these solution are different. The resulting actual twists obtained by substituting these joint-rate vectors into $V = J\dot{q}$ expressed in frame 2 coordinates are

$$ {}^2V_{s2} = {}^2J\dot{q}_{s2} = \begin{bmatrix} 1\frac{\mathrm{m}}{\mathrm{s}} \\ 2\frac{\mathrm{m}}{\mathrm{s}} \\ 0 \\ 0 \\ 0 \\ 6\frac{\mathrm{rad}}{\mathrm{s}} \end{bmatrix} \quad , \quad {}^2V_{st} = {}^tG^{2,t}\ {}^tJ\dot{q}_{st} = \begin{bmatrix} 1\frac{\mathrm{m}}{\mathrm{s}} - \frac{p_z}{2\mathrm{s}} \\ 2\frac{\mathrm{m}}{\mathrm{s}} + \frac{4p_z}{\mathrm{s}} \\ 0 \\ 0 \\ 0 \\ 6\frac{\mathrm{rad}}{\mathrm{s}} \end{bmatrix} \quad . \tag{6.143} $$

The question should then be asked, "why is solution $\dot{q}_{s2}$ better than any of the possible solutions of $\dot{q}_{st}$?" If one is to claim that there is "something special" about frame

2, for instance, such that $|^2V| \stackrel{?}{=} v_x^2 + v_y^2 + \omega_z^2$ has some useful meaning, then to get the same answer in another frame the assumed identity metric used to arrive at a physically consistent solution must be appropriately transformed in the solution for the translated frame. The identity metric used in solving for $\dot{q}_{s2}$ is $M_v = S_v^2$ of (4.26) with $\alpha_v/\alpha_\omega = \text{units}[\omega]/\text{units}[v]$ so that $V \odot M_v V$ is physically consistent. The metric in the translated frame is found from (1.84) to be

$$M_v' = ({}^2G^{t,2})^\tau \, M_v \, {}^2G^{t,2} \quad . \tag{6.144}$$

If M_v' is used in the generalized-inverse equation (4.8), then $\dot{q}_{st}' = {}^tJ^\# \; {}^tV = \dot{q}_{s2}$, *i.e.*, the generalized-inverse solution is invariant to translations and gives the same solution as that obtained using the pseudo-inverse in frame 2. (No metric M_q is needed since J has full column rank for the given configuration.)

Summarizing the results of the above example, neither the joint-rate solutions, nor the induced twists are generally equal when the pseudo-inverse solution technique is used in two frames that are translated from each other. As for all manipulators (not just planar manipulators), if the specified twist is a twist of freedom, then the solution is not dependent on the frame of calculations and the solution may then be justifiably called optimal since it is the unique solution that exactly satisfies the equation $V = J\dot{q}$.

The other two types of reciprocal manipulators, spherical and entirely prismatic-jointed, have the following decoupling characteristics. Spherical manipulators do not decouple except in the frames that have origin at the point at which all the revolute axes intersect. Entirely prismatic-jointed manipulators decouple at every point.

Finally, one other class of manipulators has been found that decouple in every frame. These are the SCARA-type manipulators. These manipulators have planar motion plus a prismatic joint perpendicular to the plane. For these manipulators

$\mathcal{W}_c = \mathcal{W}_c^z$ and a frame can always be found such that

$$[\mathcal{W}_c]_b E_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{6.145}$$

Planar and SCARA-type manipulators, which have as of now been identified as the only manipulators types that decouple in every frame, are often used by researchers to demonstrate various control algorithms. This may simplify the solutions, but may lead to invalid generalizations.

CHAPTER 7
SUMMARY AND CONCLUSIONS

Several algebraic techniques, not generally appropriate for noncommensurate systems, have been noted as being widely applied in the literature to problems in the noncommensurate system of robotics. Primary among these was the pseudo-inverse and the eigenstructure of these systems.

In linear noncommensurate systems, $u = Ax$, constraints on the possible physical units of A were given in Section 2. If these requirements are violated, the system is physically inconsistent. All linear systems can therefore be classified into either physically consistent or physically inconsistent systems. For commensurate systems, the physical units of all the elements in A are identical. The requirements on the physical units of A given for noncommensurate systems are in fact valid for all physically consistent systems.

This dissertation puts to rest the current manipulability theory. Linear noncommensurate systems do not generally have invariant or physically consistent eigenvalues and eigenvectors. The requirements for a noncommensurate system to possess an invariant eigensystem was presented in Section 2.1. In robotics, the widely accepted theory of manipulator manipulability based on the eigenstructure of various functions of the manipulator Jacobian, was shown in Chapter 5 to be invalid in all cases.

The manipulability measure, $\mathrm{Det}[J^T J]$, is valid at decouple points since this measure is physically consistent. Thus, the manipulability at a single decouple point in one configuration can be meaningfully compared to the manipulability at other configurations. But since $\mathrm{Det}[J^T J]$ is not invariant to translations, the manipulability measure at different decouple points can not be meaningfully compared.

It was also shown that physically consistent linear noncommensurate systems do not have physically consistent singular value decompositions. Only commensurate linear systems have physically consistent singular value decompositions.

The manipulator Jacobian maps possibly noncommensurate robot joint-rate vectors into noncommensurate twist vectors. The inverse velocity problem in robotics is often solved through the use of the pseudo-inverse of the Jacobian. This solution is generally arbitrary and frame dependent. It has been shown that the pseudo-inverse solution is physically inconsistent, in general, requiring the addition of elements of unlike physical units. This is due to the fact that the pseudo-inverse solution results in a minimum Euclidean-norm Euclidean least-squares solution on two generally non-Euclidean (noncommensurate) spaces. Pseudo-inverse solutions optimize physically inconsistent norms which are not invariant to either change of scale or coordinate transformations.

For some manipulators there may exist points at which the pseudo-inverse of the Jacobian is physically consistent for all frames at that point. These points are the decouple points of the manipulator. At a decouple point, the pseudo-inverse solution is invariant to scaling. Therefore, someone solving a particular problem with a coordinate system origin at a decouple point using SI units will get identical results as someone solving the same problem with a different coordinate system using British units located at the same decouple point. However, the pseudo-inverse solutions at two different decouple points are not generally the same even with identical scaling.

In robotics, the weighted generalized-inverse is known to solve the inverse velocity problem with the favorable property that the solution is frame location invariant. Metrics must be selected to force the M_v-norm of twists and the M_q-norm of joint-rates to be physically consistent. In decouple frames, the pseudo-inverse is shown to be equivalent to the generalized-inverse with identity metrics. A whole class of nonidentity metrics used with the generalized-inverse are shown to give identical solutions to the pseudo-inverse solution.

This dissertation puts to rest the arguments about the validity of the Mason/Raibert hybrid control theory of robotics and the search for a "natural decomposition" of twist and wrench spaces.

The Mason/Raibert hybrid control theory of robotics has been shown to be useful only for particular manipulators and particular choice or choices of frames. This hybrid control theory is valid when the problem is defined with respect to frames located at a decouple point. At these points the control effectively decouples the twist and wrench spaces. The hybrid control can be used at decouple points to effectively control a manipulator, although this solution is optimal with respect to a physically inconsistent norm.

It has been shown that the twist and wrench spaces of a manipulator each can be decomposed into two metric-dependent subspaces. At decouple points, the spaces can be decomposed into two subspace that are not dependent on metrics. The decomposition is accomplished with kinestatic filtering projection matrices shown in Section 6.1. A metric-dependent (generalized-inverse) hybrid control is unique with respect to the chosen metrics, and is frame independent.

In this dissertation, a class of manipulators called self-reciprocal were introduced. Planar manipulators, one type of self-reciprocal manipulator, have the peculiar property that they decouple at all points.

SCARA-type manipulators also decouple at every point. For this reason, care must be exercised in generalizing algorithms and characteristic properties of manipulators based on planar and SCARA-type manipulators. Results that are generalized from these special manipulators may prove to be invalid for manipulators that do not decouple at every point.

Several tests for determining if a manipulator has decouple points, and if so where they are located, were identified. The equivalence of these tests was discussed in Section 6.6.

Three classes of manipulators have been identified in this dissertation with respect to decouple points: manipulators that decouple at every point (all planar and SCARA-type manipulators), at a plane of points (em e.g., the SAR manipulator, at a line of points (*e.g.,* the GE P50 manipulator), at a single point (all spherical manipulators), and at no points (*e.g.,* the RPR manipulator).

Future related work includes the development of an algebra that incorporates physical units, where each elements is composed of a numerical value and a physical unit. Other algebraic properties of physically consistent noncommensurate linear systems will also be explored. And finally, further investigation of manipulators that decouple at every point will be pursued.

APPENDIX A
D-H PARAMETERS FOR VARIOUS MANIPULATORS

Denavit-Hartenberg parameters for all the various manipulators introduced in the body of this dissertation are found in the following tables along with the manipulator Jacobians expressed in various frames. For each manipulator with less than six joints, the determinants of $J^\tau J$ in various frames are given. For each manipulator with more than six joints, the determinant of JJ^τ is given.

Some Jacobians for the PR virtual manipulator of Table A.1 are

$$ {}^2J = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\tau \quad , \quad {}^tJ = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ p_y & -p_x & 0 & 0 & 0 & 1 \end{bmatrix}^\tau \quad , \tag{A.1} $$

and

$$ \mathrm{Det}[{}^2(J^\tau J)] = 1 \quad , \quad \mathrm{Det}[{}^t(J^\tau J)] = 1 + px^2 + py^2 \quad . \tag{A.2} $$

The determinant in frame 2 coordinates is physically consistent and the determinant in frame t coordinates is not physically consistent.

Some Jacobians for the RR manipulator of Table A.2 are

$$ {}^2J = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma_1 s_2 & 0 \\ c_2\sigma_1 & 0 \\ \kappa_1 & 1 \end{bmatrix} \quad , \quad {}^tJ = {}^2G_{t,2}\, {}^2J = \begin{bmatrix} p_y\kappa_1 - p_z c_2\sigma_1 & p_y \\ -p_x\kappa_1 + p_z\sigma_1 s_2 & -p_x \\ p_x c_2\sigma_1 - p_y\sigma_1 s_2 & 0 \\ \sigma_1 s_2 & 0 \\ c_2\sigma_1 & 0 \\ \kappa_1 & 1 \end{bmatrix} \quad , \tag{A.3} $$

and

$$ \mathrm{Det}[{}^2(J^\tau J)] = \sigma_1^2 \quad , \tag{A.4} $$

Table A.1. D-H parameters for PR virtual manipulator.

Joint Type	d	a	θ	α
P	d_1	0	0	0
R	d_2	0	θ_2	0

Table A.2. D-H parameters for an RR manipulator.

Joint Type	d	a	θ	α
R	0	0	θ_1	α_1
R	0	0	θ_2	0

$$\mathrm{Det}[{}^t(J^\tau J)] = \sigma_1^2(1+p_x^2+p_y^2+p_z^2)\frac{2+p_x^2+p_y^2-4p_xp_yc_2s_2+(p_x^2-p_y^2)(c_2^2-s_2^2)}{2} \quad . \tag{A.5}$$

The determinant in frame 2 coordinates is physically consistent and the determinant in frame t coordinates is not physically consistent.

Table A.3. D-H parameters for a general RRR manipulator.

Joint Type	d	a	θ	α
R	d_1	a_1	θ_1	α_1
R	d_2	a_2	θ_2	α_2
R	d_3	a_3	θ_3	α_3

A midframe Jacobian for the general RRR manipulator of Table A.3 is

$$ {}^2J = \begin{bmatrix} d_2c_2\sigma_1 + a_1\kappa_1 s_2 & 0 & 0 \\ a_2\kappa_1\kappa_2 + a_1\kappa_1\kappa_2c_2 - a_1\sigma_1\sigma_2 - a_2c_2\sigma_1\sigma_2 - d_2\kappa_2\sigma_1 s_2 & a_2\kappa_2 & 0 \\ -a_1\kappa_2\sigma_1 - a_2\kappa_2c_2\sigma_1 - a_2\kappa_1\sigma_2 - a_1\kappa_1c_2\sigma_2 + d_2\sigma_1\sigma_2s_2 & -a_2\sigma_2 & 0 \\ \sigma_1 s_2 & 0 & 0 \\ \kappa_2c_2\sigma_1 + \kappa_1\sigma_2 & \sigma_2 & 0 \\ \kappa_1\kappa_2 - c_2\sigma_1\sigma_2 & \kappa_2 & 1 \end{bmatrix} , \tag{A.6}$$

and with $\alpha_1 = \frac{\pi}{3}$, $\alpha_1 = \frac{\pi}{4}$, and $\theta_2 = \frac{\pi}{5}$,

$$\begin{aligned} \mathrm{Det}[{}^2(J^\tau J)] \quad = \quad & 0.130 + 0.5a_1^2 + 0.323a_1a_2 + 0.75a_2^2 + \\ & 0.375d_2^2 + 0.357a_2d_2 + 0.491a_2^4 + 1.213a_1a_2^3 + \\ & 0.836a_1^2a_2^2 + 0.412a_1a_2^2d_2 + 0.491a_2^2d_2^2 \quad . \end{aligned} \tag{A.7}$$

This determinant in frame 2 coordinates is not physically consistent.

Table A.4. D-H parameters for the Planar RRR manipulator.

Joint Type	d	a	θ	α
R	0	a_1	θ_1	0
R	0	a_2	θ_2	0
R	0	0	θ_3	0

Some Jacobians for the Planar RRR manipulator of Table A.4 are

$$ {}^2J = \begin{bmatrix} a_1 s_2 & 0 & 0 \\ a_2 + a_1 c_2 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad , \quad {}^tJ = {}^2G_{t,2}\, {}^2J = \begin{bmatrix} p_y + a_1 s_2 & p_y & p_y \\ a_2 - p_x + a_1 c_2 & a_2 - p_x & -p_x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad , \tag{A.8} $$

and

$$ \mathrm{Det}[{}^2(J^\tau J)] = \mathrm{Det}[{}^t(J^\tau J)] = a_1^2 a_2^2 s_2^2 \ . \tag{A.9} $$

The determinants in frame 2 and frame t coordinates are physically consistent.

Table A.5. D-H parameters for the Spherical RRR manipulator.

Joint Type	d	a	θ	α
R	0	0	θ_1	$\pi/2$
R	0	0	θ_2	$\pi/2$
R	0	0	θ_3	0

Some Jacobians for the Spherical RRR manipulator of Table A.5 are

$$ {}^2J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ s_2 & 0 & 0 \\ 0 & 1 & 0 \\ -c_2 & 0 & 1 \end{bmatrix} \quad , \quad {}^tJ = {}^2G_{t,2}\, {}^2J = \begin{bmatrix} -p_y c_2 & -p_z & p_y \\ p_x c_2 + p_z s_2 & 0 & -p_x \\ -p_y s_2 & p_x & 0 \\ s_2 & 0 & 0 \\ 0 & 1 & 0 \\ -c_2 & 0 & 1 \end{bmatrix} \quad , \tag{A.10} $$

and

$$ \mathrm{Det}[{}^2(J^\tau J)] = s_2^2 \ , \quad \mathrm{Det}[{}^t(J^\tau J)] = (1 + p_x^2 + p_y^2 + p_z^2)^2 s_2^2 \ . \tag{A.11} $$

The determinant in frame 2 coordinates is physically consistent and the determinant in frame t coordinates is not physically consistent.

Table A.6. D-H parameters for the Non-planar RRR manipulator.

Joint Type	d	a	θ	α
R	0	a_1	θ_1	$\pi/2$
R	d_2	a_2	θ_2	$\pi/2$
R	0	0	θ_3	0

Some Jacobians for the Non-planar RRR manipulator of Table A.6 are

$$ {}^1J = \begin{bmatrix} 0, 0, c_2 d_2 \\ 0, 0, d_2 s_2 \\ -a_1, 0, -a_2 \\ 0, 0, s_2 \\ 1, 0, -c_2 \\ 0, 1, 0 \end{bmatrix} \quad , \quad {}^2J = \begin{bmatrix} d_2 c_2 & 0 & 0 \\ -a_1 - a_2 c_2 & 0 & 0 \\ d_2 s_2 & -a_2 & 0 \\ s_2 & 0 & 0 \\ 0 & 1 & 0 \\ -c_2 & 0 & 1 \end{bmatrix} \quad , \tag{A.12} $$

$$ {}^3J = \begin{bmatrix} c_2 c_3 d_2 - a_1 s_3 - a_2 c_2 s_3 & 0 & 0 \\ -a_1 c_3 - a_2 c_2 c_3 - c_2 d_2 s_3 & 0 & 0 \\ d_2 s_2 & -a_2 & 0 \\ c_3 s_2 & s_3 & 0 \\ -s_2 s_3 & c_3 & 0 \\ -c_2 & 0 & 1 \end{bmatrix} \quad , \tag{A.13} $$

and

$$ \mathrm{Det}[{}^1(J^\tau J)] = s_2^2 + a_1^2 + a_2^2 + d_2^2 + 2a_1 a_2 c_2 + a_1^2 d_2^2 \quad , \tag{A.14} $$

$$ \begin{aligned} \mathrm{Det}[{}^2(J^\tau J)] = \mathrm{Det}[{}^3(J^\tau J)] &= s_2^2 + a_1^2 + a_2^2 + d_2^2 + 2a_1 a_2 c_2 \\ &\quad + a_2^2 \left[a_1^2 + 2a_1 a_2 c_2 + c_2^2 (a_2^2 + d_2^2) \right] \quad . \end{aligned} \tag{A.15} $$

The determinants in frame 1, 2, and 3 coordinates are not physically consistent.

Table A.7. D-H parameters for the PPP orthogonal manipulator.

Joint Type	d	a	θ	α
P	d_1	0	0	$-\pi/2$
P	d_2	0	$\pi/2$	$\pi/2$
P	d_3	0	0	0

Some Jacobians for the PPP Orthogonal manipulator of Table A.7 are

$$ {}^0J = {}^tJ = {}^0G_{t,0}\, {}^0J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad {}^2J = {}^3J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \tag{A.16} $$

and

$$\mathrm{Det}[{}^{g}(J^{\tau}J)] = 1 \quad , \tag{A.17}$$

for any frame g. The determinant in any frame coordinates is physically consistent.

Table A.8. D-H parameters for the PRP Small Assembly Robot (SAR).

Joint Type	d	a	θ	α
P	d_1	0	0	0
R	0	0	θ_2	$\pi/2$
P	d_3	0	0	0

Some Jacobians for the SAR manipulator of Table A.8 are

$${}^{2}J = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad {}^{3}J = \begin{bmatrix} 0 & d_3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \tag{A.18}$$

and

$$\mathrm{Det}[{}^{2}(J^{\tau}J)] = 1 \quad , \quad \mathrm{Det}[{}^{3}(J^{\tau}J)] = 1 + d_3^2 \quad . \tag{A.19}$$

The determinant in frame 2 coordinates is physically consistent and the determinant in frame 3 coordinates is not physically consistent.

Table A.9. D-H parameters for the RPR manipulator.

Joint Type	d	a	θ	α
R	0	0	θ_1	$\pi/2$
P	d_2	0	$\pi/2$	$\pi/2$
R	0	0	θ_3	0

Some Jacobians for the RPR manipulator of Table A.9 are

$${}^{2}J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ d_2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad {}^{t}J = {}^{2}G_{t,2}\,{}^{2}J = \begin{bmatrix} 0 & 0 & p_y \\ p_z & 1 & -p_x \\ d_2 - p_y & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \tag{A.20}$$

Table A.10. D-H parameters for the RRRP-1 SCARA manipulator.

Joint Type	d	a	θ	α
R	0	a_1	θ_1	0
R	0	a_2	θ_2	0
R	0	0	θ_3	0
P	d_4	0	0	0

and

$$\mathrm{Det}[^2(J^\tau J)] = 1 + d_2^2 \quad , \quad \mathrm{Det}[^t(J^\tau J)] = (1+p_y^2)(1+d_2^2 - 2d_2p_y + p_y^2) \quad . \tag{A.21}$$

The determinant in frame 2 coordinates is physically consistent and the determinant in frame t coordinates is not physically consistent.

Some Jacobians for the SCARA manipulator of Table A.10 are

$$^2J = \begin{bmatrix} a_1s_2 & 0 & 0 & 0 \\ a_2+a_1c_2 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad , \tag{A.22}$$

$$^tJ = {}^2G_{t,2}\,{}^2J = \begin{bmatrix} \frac{1}{a_1s_2} & 0 & 0 & 0 & 0 & \frac{-p_y}{a_1s_2} \\ -\frac{a_2+a_1c_2}{a_1a_2s_2} & \frac{1}{a_2} & 0 & 0 & 0 & \frac{a_2p_y+a_1c_2p_y+a_1p_xs_2}{a_1a_2s_2} \\ \frac{c_2}{a_2s_2} & \frac{-1}{a_2} & 0 & 0 & 0 & \frac{-c_2p_y+a_2s_2-s_2p_x}{a_2s_2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad , \tag{A.23}$$

and

$$\mathrm{Det}[^2(J^\tau J)] = \mathrm{Det}[^t(J^\tau J)] = (a_1a_2s_2)^2 \quad . \tag{A.24}$$

The determinants in frame 2 and frame t coordinates are physically consistent.

Table A.11. D-H parameters for the RRRP-2 manipulator.

Joint Type	d	a	θ	α
R	0	a_1	θ_1	$\pi/2$
R	0	a_2	θ_2	0
R	0	0	θ_3	$\pi/2$
P	d_4	0	0	0

Some Jacobians for the RRRP-2 manipulator of Table A.11 are

$$
{}^0J = \begin{bmatrix} 0 & 0 & a_2c_1s_2 & c_1s_{2+3} \\ 0 & 0 & a_2s_1s_2 & s_1s_{2+3} \\ 0 & -a_1 & -a_1 - a_2c_2 & -c_{2+3} \\ 0 & s_1 & s_1 & 0 \\ 0 & -c_1 & -c_1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \ , \quad {}^2J = \begin{bmatrix} 0 & 0 & 0 & s_3 \\ 0 & a_2 & 0 & -c_3 \\ -a_1 - a_2c_2 & 0 & 0 & 0 \\ s_2 & 0 & 0 & 0 \\ c_2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \ , \tag{A.25}
$$

and

$$
\mathrm{Det}[{}^0(J^\tau J)] = a_2^2 s_3^2 \ , \tag{A.26}
$$

$$
\mathrm{Det}[{}^2(J^\tau J)] = a_2^2 s_3^2 + a_1^2 a_2^2 s_3^2 + 2a_1 a_2^3 c_2 s_3^2 + a_2^4 c_2^2 s_3^2 \ . \tag{A.27}
$$

The determinant in frame 0 coordinates is physically consistent and the determinant in frame 2 coordinates is not physically consistent.

Table A.12. D-H parameters for the RRRP-3 manipulator.

Joint Type	d	a	θ	α
R	0	0	θ_1	$\pi/2$
R	0	a_2	θ_2	$\pi/2$
R	0	a_3	θ_3	$\pi/2$
P	d_4	0	0	0

Some Jacobians for the RRRP-3 manipulator of Table A.12 are

$$
{}^1J = \begin{bmatrix} 0 & 0 & 0 & c_2s_3 \\ 0 & 0 & 0 & s_2s_3 \\ 0 & 0 & -a_2 & -c_3 \\ 0 & 0 & s_2 & 0 \\ 1 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \ , \quad {}^2J = \begin{bmatrix} 0 & 0 & 0 & s_3 \\ -a_2c_2 & 0 & 0 & -c_3 \\ 0 & -a_2 & 0 & 0 \\ s_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -c_2 & 0 & 1 & 0 \end{bmatrix} \ , \tag{A.28}
$$

and

$$
\mathrm{Det}[{}^1(J^\tau J)] = c_3^2 s_2^2 + c_2^2 s_2^2 s_3^2 + s_2^4 s_3^2 + a_2^2 s_3^2 \ , \tag{A.29}
$$

$$
\mathrm{Det}[{}^2(J^\tau J)] = s_2^2 + a_2^2 s_2^2 + a_2^2 c_2^2 s_3^2 + a_2^4 c_2^2 s_3^2 \ . \tag{A.30}
$$

The determinants in frame 1 and 2 coordinates are not physically consistent.

Table A.13. D-H parameters for 5R GE-P50 manipulator.

Joint Type	d	a	θ	α
R	0	0	θ_1	$\pi/2$
R	a_2	0	θ_2	0
R	a_3	0	θ_3	0
R	0	0	θ_4	$\pi/2$
R	0	0	θ_5	0

Some Jacobians for the GE-P50 manipulator of Table A.13 are

$$
{}^2J = \begin{bmatrix} 0 & 0 & 0 & a_3 s_3 & 0 \\ 0 & a_2 & 0 & -a_3 c_3 & 0 \\ -a_2 c_2 & 0 & 0 & 0 & -a_3 c_4 \\ s_2 & 0 & 0 & 0 & s_{3+4} \\ c_2 & 0 & 0 & 0 & -c_{3+4} \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} , \tag{A.31}
$$

$$
{}^3J = \begin{bmatrix} 0 & a_2 s_3 & 0 & 0 & 0 \\ 0 & a_3 + a_2 c_3 & a_3 & 0 & 0 \\ -a_2 c_2 - a_3 c_{2+3} & 0 & 0 & 0 & 0 \\ s_{2+3} & 0 & 0 & 0 & s_4 \\ c_{2+3} & 0 & 0 & 0 & -c_4 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} , \tag{A.32}
$$

and

$$
\mathrm{Det}[{}^2(J^\tau J)] = a_2^2 a_3^2 s_3^2 (a_2^2 c_2^2 + 2 a_2 a_3 c_2^2 c_{3+4} c_4 + a_3^2 c_4^2 + s_{2+3+4}^2) \ , \tag{A.33}
$$

$$
\begin{aligned} \mathrm{Det}[{}^3(J^\tau J)] \quad &= \quad a_2^2 a_3^2 s_3^2 \left[a_2^2 c_2^2 + a_2 a_3 c_3 + 2 a_3^2 c_{2+3}^2 + \right. \\ & \qquad \left. a_2 a_3 (c_2 c_{2+3} - s_2 s_{2+3}) + s_{2+3+4}^2 \right] \ . \end{aligned} \tag{A.34}
$$

In a frame t translated from frame 3 by

$$
p = \left[\frac{(a_2 c_2 + a_3 c_{2+3}) s_4}{s_{2+3+4}}, \ -\frac{(a_2 c_2 + a_3 c_{2+3}) c_4}{s_{2+3+4}}, \ 0 \right]^\tau \ , \tag{A.35}
$$

the Jacobian, ${}^tJ = {}^3G_{t,3}\, {}^3J$, is

$$
{}^tJ = \begin{bmatrix} 0 & -\frac{(c_{2+3}(a_2 c_{3+4} + a_3 c_4)}{s_{2+3+4}} & -\frac{(a_2 c_2 + a_3 c_{2+3}) c_4}{s_{2+3+4}} & -\frac{(a_2 c_2 + a_3 c_{2+3}) c_4}{s_{2+3+4}} & 0 \\ 0 & \frac{(a_2 c_{3+4} + a_3 c_4) s_{2+3}}{s_{2+3+4}} & a_3 - \frac{(a_2 c_2 + a_3 c_{2+3}) s_4}{s_{2+3+4}} & -\frac{(a_2 c_2 + a_3 c_{2+3}) s_4}{s_{2+3+4}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ s_{2+3} & 0 & 0 & 0 & s_4 \\ c_{2+3} & 0 & 0 & 0 & -c_4 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} , \tag{A.36}
$$

and

$$\text{Det}[{}^{t}(J^{\tau}J)] = a_2^2 a_3^2 s_3^2 s_{2+3+4}^2 \quad . \tag{A.37}$$

The determinants in frame 2 and 3 coordinates are not physically consistent and the determinant in frame t coordinates is physically consistent.

Table A.14. D-H parameters for the 7R Redundant Anthropomorphic Arm.

Joint Type	d	a	θ	α
R	0	0	θ_1	$\pi/2$
R	0	0	θ_2	$\pi/2$
R	0	a_3	θ_3	$\pi/2$
R	0	0	θ_4	$\pi/2$
R	0	a_5	θ_5	$\pi/2$
R	0	0	θ_6	$\pi/2$
R	0	0	θ_7	0

The 7R Redundant Anthropomorphic Arm manipulator of Table A.14 has three intersecting shoulder joints, an elbow joint, and three intersecting wrist joints. A midframe Jacobian for this manipulator is

$$ {}^{4}J = \begin{bmatrix} a_3 s_2 s_3 s_4 & -a_3 c_3 s_4 & 0 & 0 & 0 & 0 & -a_5 c_6 s_5 \\ a_3 c_2 & 0 & -a_3 & 0 & 0 & 0 & a_5 c_5 c_6 \\ -a_3 c_4 s_2 s_3 & a_3 c_3 c_4 & 0 & 0 & 0 & -a_5 & 0 \\ c_3 c_4 s_2 - c_2 s_4 & c_4 s_3 & s_4 & 0 & 0 & s_5 & c_5 s_6 \\ s_2 s_3 & -c_3 & 0 & 1 & 0 & -c_5 & s_5 s_6 \\ c_2 c_4 + c_3 s_2 s_4 & s_3 s_4 & -c_4 & 0 & 1 & 0 & -c_6 \end{bmatrix} \quad . \tag{A.38}$$

When $\theta_1 = 1\text{rad}$, $\theta_2 = 2\text{rad}$, $\theta_3 = 3\text{rad}$, $\theta_4 = 4\text{rad}$, $\theta_5 = 5\text{rad}$, $\theta_6 = 6\text{rad}$, and $\theta_7 = 7\text{rad}$,

$$\text{Det}[JJ^{\tau}] = 0.8041 a_3^4 a_5^2 + 0.2279 a_3^3 a_5^3 + 0.6646 a_3^2 a_5^4 \quad . \tag{A.39}$$

This determinant is physically consistent.

Table A.15. D-H parameters for the 7R CESAR Research Manipulator.

Joint Type	d	a	θ	α
R	0	0	θ_1	$\pi/2$
R	d_2	0	θ_2	$-\pi/2$
R	$-d_3$	a_3	θ_3	$\pi/2$
R	0	a_4	θ_4	0
R	0	0	θ_5	$-\pi/2$
R	0	0	θ_6	$\pi/2$
R	d_7	0	θ_7	0

A midframe Jacobian for the 7R CESAR Research Manipulator [21] of Table A.15 is

$$
{}^3J = \begin{bmatrix}
c_2c_3d_2 + d_3s_2s_3 & c_3d_3 & 0 & 0 & a_4s_4 & 0 & a_4c_6s_4 \\
-d_2s_2 + a_3s_2s_3 & a_3c_3 & 0 & 0 & -a_4c_4 & 0 & -a_4c_4c_6 \\
-a_3c_2 - c_3d_3s_2 + c_2d_2s_3 & d_3s_3 & -a_3 & 0 & 0 & a_4c_5 & a_4s_5s_6 \\
c_3s_2 & -s_3 & 0 & 0 & 0 & -s_{4+5} & c_{4+5}s_6 \\
c_2 & 0 & 1 & 0 & 0 & c_{4+5} & s_{4+5}s_6 \\
s_2s_3 & c_3 & 0 & 1 & 1 & 0 & c_6
\end{bmatrix} . \tag{A.40}
$$

When $\theta_1 = 1\text{rad}$, $\theta_2 = 2\text{rad}$, $\theta_3 = 3\text{rad}$, $\theta_4 = 4\text{rad}$, $\theta_5 = 5\text{rad}$, $\theta_6 = 6\text{rad}$, $\theta_7 = 7\text{rad}$, $a_3 = a_4 = \beta$, and $d_2 = d_3 = \gamma$,

$$
\text{Det}[JJ^\tau] = 0.4056\beta^6 + 1.1376\beta^5\gamma + 1.1679\beta^4\gamma^2 + 0.5292\beta^3\gamma^3 + 0.0969\beta^2\gamma^4 \ . \tag{A.41}
$$

This determinant is physically consistent.

Table A.16. D-H parameters for the 7R K-1207 Robot Research Arm.

Joint Type	d	a	θ	α
R	0	a_1	θ_1	$-\pi/2$
R	0	a_2	θ_2	$\pi/2$
R	d_3	a_3	θ_3	$-\pi/2$
R	0	a_4	θ_4	$\pi/2$
R	d_5	a_5	θ_5	$-\pi/2$
R	0	a_6	θ_6	$\pi/2$
R	0	0	θ_7	0

The 7R K-1207 Robot Research Arm [33] of Table A.16 is functionally equivalent (with certain α's and d's set to zero) to the manipulator described in [50] and to the

UJIBOT in [43]. A midframe Jacobian for this manipulator is

$$
{}^3J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -a_5s_4 & -a_5c_4c_6s_5 \\ a_3s_2s_3 & -a_3c_3 & 0 & 0 & 0 & a_5c_4 & -a_5c_6s_4s_5 \\ a_3c_2 & 0 & -a_3 & 0 & 0 & 0 & a_5c_5c_6 \\ c_3s_2 & s_3 & 0 & 0 & s_4 & c_4s_5 & -c_6s_4+c_4c_5s_6 \\ -c_2 & 0 & 1 & 0 & -c_4 & s_4s_5 & c_4c_6+c_5s_4s_6 \\ s_2s_3 & -c_3 & 0 & 1 & 0 & -c_5 & s_5s_6 \end{bmatrix} . \tag{A.42}
$$

When $\theta_1 = 1\text{rad}$, $\theta_2 = 2\text{rad}$, $\theta_3 = 3\text{rad}$, $\theta_4 = 4\text{rad}$, $\theta_5 = 5\text{rad}$, $\theta_6 = 6\text{rad}$, $\theta_7 = 7\text{rad}$, $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = \beta$, and $d_3 = d_5 = \gamma$,

$$
\begin{aligned} \mathrm{Det}[JJ^\tau] = \quad & 6.0495\beta^6 + 14.0083\beta^5\gamma + 13.4816\beta^4\gamma^2 + 4.3543\beta^3\gamma^3 + \\ & 5.5013\beta^2\gamma^4 + 1.4809\beta\gamma^5 + 0.1103\gamma^6 \ . \end{aligned} \tag{A.43}
$$

This determinant is physically consistent.

Table A.17. D-H parameters for the 7R PUMA-260+1 Spherical Wrist Manipulator.

Joint Type	d	a	θ	α
R	0	0	θ_1	$\pi/2$
R	0	a_2	θ_2	0
R	d_3	0	θ_3	$\pi/2$
R	d_4	0	θ_4	$\pi/2$
R	0	0	θ_5	$\pi/2$
R	0	0	θ_6	$\pi/2$
R	0	0	θ_7	0

A midframe Jacobian for the 7R PUMA-260+1 Spherical Wrist Manipulator of Table A.17 is

$$
{}^4J = \begin{bmatrix} c_{2+3}c_4d_3 - a_2c_2s_4 - d_4s_{2+3}s_4 & c_4d_4 + a_2c_4s_3 & c_4d_4 & 0 & 0 & 0 & 0 \\ d_3s_{2+3} & -a_2c_3 & 0 & 0 & 0 & 0 & 0 \\ a_2c_2c_4 + c_4d_4s_{2+3} + c_{2+3}d_3s_4 & d_4s_4 + a_2s_3s_4 & d_4s_4 & 0 & 0 & 0 & 0 \\ c_4s_{2+3} & s_4 & s_4 & 0 & 0 & s_5 & c_5s_6 \\ -c_{2+3} & 0 & 0 & 1 & 0 & -c_5 & s_5s_6 \\ s_{2+3}s_4 & -c_4 & -c_4 & 0 & 1 & 0 & -c_6 \end{bmatrix} , \tag{A.44}
$$

$$
\begin{aligned} \mathrm{Det}[JJ^\tau] \ = \ & a_2^2c_3^2d_4^2(a_2c_2 + d_4s_{2+3})^2 \Big(1 + 2s_5^2 + 2s_6^2 \\ & + c_5^2c_6^2 - s_5^2s_6^2 + s_5^2c_6^2 + c_5^2s_6^2\Big)/2 \ . \end{aligned} \tag{A.45}
$$

Table A.18. D-H parameters for the 3P-4R Redundant Spherical Wrist Robot.

Joint Type	d	a	θ	α
P	d_1	0	0	$\pi/2$
P	d_2	0	$\pi/2$	$\pi/2$
P	d_3	0	0	0
R	0	0	θ_4	$\pi/2$
R	0	0	θ_5	$\pi/2$
R	0	0	θ_6	$\pi/2$
R	0	0	θ_7	0

This determinant is physically consistent.

A midframe Jacobian for the 3P-4R Redundant Spherical Wrist Robot of Table A.18 is

$$ {}^4J = \begin{bmatrix} c_4 & s_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ s_4 & -c_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_5 & c_5 s_6 \\ 0 & 0 & 0 & 1 & 0 & -c_5 & s_5 s_6 \\ 0 & 0 & 0 & 0 & 1 & 0 & -c_6 \end{bmatrix} , \tag{A.46} $$

and

$$ \mathrm{Det}[JJ^{\tau}] = \left(1 + 2s_5^2 + 2s_6^2 - c_5^2 c_6^2 - s_5^2 s_6^2 + s_5^2 c_6^2 + c_5^2 s_6^2\right)/2 \ . \tag{A.47} $$

This determinant is physically consistent.

Table A.19. D-H parameters for the 2R-P-4R GP66+1 Manipulator.

Joint Type	d	a	θ	α
R	0	0	θ_1	$\pi/2$
R	0	a_2	θ_2	$\pi/2$
P	d_3	0	0	0
R	0	0	θ_4	$\pi/2$
R	d_5	0	θ_5	$\pi/2$
R	0	0	θ_6	$\pi/2$
R	0	0	θ_7	0

A midframe Jacobian for the (2R-P-4R) GP66+1 Manipulator of Table A.19 is

$$ {}^4J = \begin{bmatrix} -a_2 c_2 s_4 - d_3 s_2 s_4 & c_4 d_3 & 0 & 0 & 0 & c_5 d_5 & -d_5 s_5 s_6 \\ 0 & -a_2 & 1 & 0 & 0 & d_5 s_5 & c_5 d_5 s_6 \\ a_2 c_2 c_4 + c_4 d_3 s_2 & d_3 s_4 & 0 & 0 & 0 & 0 & 0 \\ c_4 s_2 & s_4 & 0 & 0 & 0 & s_5 & c_5 s_6 \\ -c_2 & 0 & 0 & 1 & 0 & -c_5 & s_5 s_6 \\ s_2 s_4 & -c_4 & 0 & 0 & 1 & 0 & -c_6 \end{bmatrix} . \tag{A.48} $$

When $\theta_1 = 1\text{rad}$, $\theta_2 = 2\text{rad}$, $\theta_3 = 3\text{rad}$, $\theta_4 = 4\text{rad}$, $\theta_5 = 5\text{rad}$, $\theta_6 = 6\text{rad}$, $\theta_7 = 7\text{rad}$, and $a_2 = d_5 = \beta$,

$$\begin{aligned}\text{Det}[JJ^T] = {} & 0.0234\beta^4 + 0.0450\beta^3 d_3 - 0.0209\beta^2 d_3^2 - 0.9312\beta d_3^3 + 1.531 d_3^4 \\ & +0.000342\beta^6 - 0.00614\beta^5 d_3 + 0.0276\beta^4 d_3^2 \ . \end{aligned} \tag{A.49}$$

This determinant is not physically consistent.

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BIOGRAPHICAL SKETCH

Eric Michael Schwartz was born in Miami, Florida, on April 20, 1959. In 1977 he graduated from Southwest Miami Senior High School in the top one percent of his class. He received two Bachelor of Science degrees from the University of Florida in April 1984 (both with high honors)—one in electrical engineering and the other in mechanical engineering. He received a Master of Engineering degree from the University of Florida in August 1989.

Interspersed with his undergraduate and graduate school career, he has worked at various jobs in the engineering industry, holding positions with Bendix Avionics, Universal Securities Instruments, IBM, the Allen-Bradley Company, and the Electronics Communications Laboratory at the University of Florida. He has also worked at the University of Florida as a teaching assistant and a research assistant, and was an instructor for two electrical engineering courses. He taught a short robotics laboratory course at Technopolis Institute in Bari, Italy. He has been a member of Professor Keith L. Doty's Machine Intelligence Laboratory at the University of Florida since 1985.

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